On the Approximations of Multiple target filtering equations

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∼ Three joint articles with F. Caron, A. Doucet, M. Pace, B.N. Vo

• **Particle approximations of a class of branching distribution flows arising in multi-target tracking. Joint work with Fr. Caron, A. Doucet, and M. Pace HAL-INRIA RR-7233 (2010).**
Under minor revision
• **On the Conditional Distributions of Spatial Point Processes. Joint work with Fr. Caron, A. Doucet, and M. Pace.** [HAL-INRIA RR-7232 (2010)].
To appear in the Advances in Applied Probability (2011).
• **On the Stability & the Approximation of Branching Distribution Flows, with Applications to Nonlinear Multiple Target Filtering. Joint work with Fr. Caron, M. Pace, and B.N. Vo.** HAL-INRIA RR-7376 (2010).
To appear in Stochastic Analysis and Applications (2011).

Summary

Some notation : E measurable space

$(\mathcal{M}(E), \mathcal{P}(E), \mathcal{B}(E))=$ (measures, probabilities, bounded functions) on E.

- $\mu(f) = \int \mu(dx)\ f(x) \quad \text{and} \quad \overline{\mu}(dx) := \mu(dx)/\mu(1) \in \mathcal{P}(E)$
- $Q(x,dy)$ **integral operator over E** (composition $(Q_1 Q_2)$)

$$\begin{aligned} Q(f)(x) &= \int Q(x,dy) f(y) \\ [\mu Q](dy) &= \int \mu(dx) Q(x,dy) \qquad (\Longrightarrow [\mu Q](f) = \gamma[Q(f)]\ ) \end{aligned}$$

- **Boltzmann-Gibbs transformation :** $G : E \to [0,\infty[$ with $\mu(G) > 0$

$$\Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx) \quad \exists \text{ Markov transport} \quad \Psi_G(\mu) = \mu S_\mu$$

Example $(G \leq 1) \rightsquigarrow$ accept/reject/interacting jump transition

$$S_\mu(x,dy) = G(x)\delta_x(dy) + (1 - G(x))\ \Psi_G(\mu)(dy)$$

Summary

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- 3 simple ingredients :

  Potential $G_n(x) \geq 1$, measure $\mu_n(dx) \geq 0$ and $M_n(x_{n-1}, dx_n)$ Markov.

  - **Branching rule (spawning)** :

    $$x \rightsquigarrow g_n(x) \text{ offsprings, with } \mathbb{E}(g_n(x)) = G_n(x)$$

    $\supset$ *survival rates* $e_n(x)$ + *cemetery states :* $G_n \rightsquigarrow e_n(x)G_n(x)$
  - **Spontaneous births:** Spatial Poisson with intensity $\mu_n(dx)$
  - **Free motion between branching times :** M_n-evolutions

- $\rightsquigarrow$ Random occupation measure (after the n-th evolution step)

$$\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$$

Spatial Branching models (time index $n \in \mathbb{N}$, state spaces E_n)

- First moment recursion = branching intensity distribution

$$\begin{aligned}\gamma_{n+1}(f) &:= \mathbb{E}\left(\mathcal{X}_{n+1}(f)\right)\\ &= \gamma_n(Q_{n+1}(f)) + \mu_{n+1}(f) \text{ with } Q_{n+1}(x,dy) = G_n(x) M_{n+1}(x,dy)\end{aligned}$$

Sketched proof for $\mu_n = 0$:

$$\mathcal{X}_{n+1} = \sum_{i=1}^{N_{n+1}} \delta_{X^i_{n+1}} = \sum_{i=1}^{N_n} \sum_{j=1}^{g^i_n(X^i_n)} \delta_{X^{i,j}_{n+1}}$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n, g_n(X_n)\right) = \sum_{i=1}^{N_{n-1}} g^i_n(X^i_n)\ M_{n+1}(f)(X^i_n)$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathcal{X}_{n+1}(f) \mid \mathcal{X}_n\right) = \sum_{i=1}^{N_{n-1}} G_n(X^i_n)\ M_{n+1}(f)(X^i_n) = \mathcal{X}_n(G_n M_{n+1}(f))$$

Summary

$\mu_n = 0 \Rightarrow$ Conventional Feynman-Kac models

- First moment recursion

$$\gamma_{n+1}(f) = \gamma_n(Q_{n+1}(f)) = \gamma_{n-1}(Q_n\,(Q_{n+1}(f))) = \ldots$$

- Feynman-Kac representation ($\supset \uparrow$ Application domains)

$$\gamma_{n+1}(f) \;=\; \gamma_0(1)\;\mathbb{E}_{\eta_0}\left(f(X_{n+1}) \prod_{0 \le p \le n} G_p(X_p) \right)$$

- Particle approximations = Genetic type algorithms = Particle filters = ...

$$Q_{n+1}(x, dy) = \underbrace{G_n(x)}_{\text{Selection potential}} \times \underbrace{M_{n+1}(x, y)}_{\text{Mutation transition}}$$

Summary

More general spatial Branching models

$$\gamma_n = \gamma_{n-1} Q_n + \mu_n \quad \text{and} \quad \eta_n := \gamma_n/\gamma_n(1) \quad (\text{\& hypothesis:} \quad \gamma_0 = \mu_0)$$

$$\Updownarrow$$

$$(\gamma_n(1), \eta_n) = \Gamma_n\left(\gamma_{n-1}(1), \eta_{n-1}\right) := \Gamma_{p,n}\left(\gamma_p(1), \eta_p\right) \quad \text{Nonlinear semigroup}$$

Some problems

- Problem 1: Mass process $\gamma_n(1)$ "unstable" $\gamma_n(1) \uparrow \infty$ or $\gamma_n(1) \downarrow 0$ as $n \uparrow \infty$
- Problem 2: $\mathcal{X}_n = \sum_{i=1}^{N_n} \delta_{X_n^i}$ generally NOT POISSON random field.
- Problem 3: $\exists$ non generate numerical sampling method?
- Problem 4: $\exists$ non generate approximation of γ_n?

Some answers:

$\supset$ Particle approximations of a class of branching distribution flows arising in multi-target tracking. Joint work with Fr. Caron, A. Doucet, and M. Pace HAL-INRIA RR-7233 (March 2010).

Three scenarios $\gamma_n = \gamma_{n-1} Q + \mu$ with $Q(x,.) = G(x)M(x,.)$

$$g_- := \inf G \leq G = Q(1) \leq \sup G := g_+$$

- $G = 1$ & $\eta_\infty := \eta_\infty M$

$$\gamma_n(1) = \gamma_0(1) + n\mu(1) \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\rm tv} = O\left(1/n\right)$$

- $g_+ < 1$ & $\eta_\infty := \gamma_\infty / \gamma_\infty(1)$ with $\gamma_\infty := \sum_{n\geq 0} \mu Q^n$ (when $\gamma_0 = \mu$)

$$|\gamma_n(f) - \gamma_\infty(f)| \vee |\eta_n(f) - \eta_\infty(f)| \leq c\ g_+^n\ \|f\|$$

- $g_- > 1$ & $\eta_\infty(f) := \eta_\infty Q(f)/\eta_\infty Q(1)$

  $\eta_\infty =$
  [quasi-invariant measures, Yaglom measures, ground states, Feynman-Kac semigroup fixed points, infinite population stationary measure, etc.]

$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_n(1) = \log \eta_\infty(G) \quad \text{and} \quad \|\eta_n - \eta_\infty\|_{\rm tv} \leq c\ e^{-\lambda n}$$

Nonlinear equations

$$\eta_{n+1}(f) \propto \gamma_n(1)\ \eta_n(Q_{n+1}(f)) + \mu_{n+1}(1)\ \overline{\mu}_{n+1}(f)$$

$$\Downarrow$$

Nonlinear & interacting mass + proba measures equations

$$\left\{\begin{array}{rcl}\gamma_{n+1}(1) & = & \gamma_n(1)\ \eta_n(G_n) + \mu_{n+1}(1) \\ \\ \eta_{n+1} & = & \Psi_{G_n}(\eta_n) M_{n+1,(\gamma_n(1),\eta_n)}\end{array}\right.$$

with the Markov transitions:

$$M_{n+1,(m,\eta)}(x,dy) := \alpha_n(m,\eta)\, M_{n+1}(x,dy) + (1-\alpha_n(m,\eta))\ \overline{\mu}_{n+1}(dy)$$

with the collection of $[0,1]$-parameters

$$\alpha_n(m,\eta) = \frac{m\ \eta(G_n)}{m\ \eta(G_n) + \mu_{n+1}(1)}$$

A nonlinear filtering/Feynman-Kac type model

$$\eta_n \xrightarrow{\text{updating}} \widehat{\eta}_n := \Psi_{G_n}(\eta_n) = \eta_n S_{n,\eta_n} \xrightarrow{\text{prediction}} \eta_{n+1} := \widehat{\eta}_n M_{n+1,(\gamma_n(1),\eta_n)}$$

$$\Downarrow$$

A couple of equations:

- the total mass evolution

$$\gamma_{n+1}(1) = \gamma_n(1)\ \eta_n(G_n) + \mu_{n+1}(1)$$

- The "nonlinear filtering/Feynman-Kac type" conservative equations

$$\eta_{n+1} = \eta_n S_{n,\eta_n} M_{n+1,(\gamma_n(1),\eta_n)} := \eta_n \underbrace{K_{n+1,(\gamma_n(1),\eta_n)}}_{\textbf{Markov transition}}$$

Summary

Mean field interacting particle models

$$\eta_n^N = \frac{1}{N} \sum_{1 \le i \le N} \delta_{\xi_n^{(N,i)}} \simeq_{N\uparrow\infty} \eta_n \quad \text{and} \quad \gamma_n^N(1) \simeq_{N\uparrow\infty} \gamma_n(1)$$

Genetic type population evolution :

- the total mass evolution **["deterministic"]**

$$\gamma_{n+1}^N(1) := \gamma_n^N(1)\ \eta_n^N(G_n) + \mu_{n+1}(1)$$

- Mean field particle model

$$\forall 1 \le i \le N \quad \xi_{n+1}^{(N,i)} = \text{random var. with law} \quad K_{n+1,(\gamma_n^N(1),\eta_n^N)}(\xi_n^{(N,i)}, dx_{n+1})$$

$$\Downarrow$$

with sampling error fluctuations:

$$\eta_{n+1}^N := \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)} + \frac{1}{\sqrt{N}}\ W_{n+1}^N$$

Theoretical convergence results

- **Independent local sampling error fluctuations**

$$(W_n^N)_{n\geq 0} \simeq_{N\uparrow\infty} \quad \text{iid centered Gaussian fields } (W_n)_{n\geq 0}$$

- **Functional CLT(s) (with** $\left[\gamma_n^N := \gamma_n^N(1) \times \eta_n^N\right]$**)**

$$V_n^{\gamma,N} := \sqrt{N}\,(\gamma_n^N - \gamma_n) \quad \& \quad V_n^{\eta,N} := \sqrt{N}\,(\eta_n^N - \eta_n) \quad \to_N \quad V_n^\gamma \quad \& \quad V_n^\eta$$

- **Uniform convergence results (under some mixing conditions on** M_n**)**

$$\sup_{n\geq 0} \mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right|^p\right) \leq c(p)/N^{p/2} \quad (\oplus \text{ uniform expo. concentration})$$

- **Unbiased particle total mass with variance**

$$N\,\mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \leq c\;n\;\left(1 + \frac{c}{N}\right)^{n-1}$$

Conditioning principles for marked point processes

- **Poisson point process** $\mathcal{X}$ with intensity $\gamma(dx_1)\ Q(x_1, dx_2)$ on $E = (E_1 \times E_2)$

$$\mathcal{X} := m_N(X_1, X_2) = \sum_{1 \leq i \leq N} \delta_{(X_1^i, X_2^i)} \quad \text{and} \quad \mathcal{X}_j := m_N(X_j) = \sum_{1 \leq i \leq N} \delta_{X_j^i}$$

- **2 Bayes' rules:**

Normalization $p(x_2|x_1)$ $\oplus$ Reversal Markov operator $p(x_1|x_2)$

$$\overline{Q}(x_1, dx_2) = \frac{Q(x_1, dx_2)}{Q(x_1, E_2)} \quad \text{and} \quad \gamma(dx_1)\ Q(x_1, dx_2) = (\gamma Q)\,(dx_2)\ Q_\gamma(x_2, dx_1)$$

- $\Rightarrow$ **2 conditional distributions formulae:**

$$\mathbb{E}\left(F_1(\mathcal{X}_1)\ \mid \mathcal{X}_2\right) = \int F_1\left(m_N(x_1)\right) \prod_{1 \leq i \leq N} Q_\gamma(X_2^i, dx_1^i)$$

$$\mathbb{E}\left(F_2(\mathcal{X}_2)\ \mid \mathcal{X}_1\right) = \int F_2\left(m_N(x_2)\right) \prod_{1 \leq i \leq N} \overline{Q}(X_1^i, dx_2^i)$$

Conditioning principles for marked point processes

- $(\mathcal{X}_1, \mathcal{X}_2) = (\mathcal{X}, \mathcal{Y})$, $F(\mathcal{X}) = \mathcal{X}(f)$ and $P(y, dx)$ Markov transition

$$\begin{aligned}\widehat{\gamma}(f) &:= \mathbb{E}\left(F(\mathcal{X}) \mid \mathcal{Y}\right) \\ &= \int F\left(m_N(x)\right) \prod_{1 \leq i \leq N} P(Y^i, dx^i) = \mathcal{Y}(P(f))\end{aligned}$$

- **Example with $\mathcal{X}$ Poisson Signal $\gamma(dx) \rightsquigarrow \mathcal{Y}$ Poisson Observation**

$$(X^i = x) \rightsquigarrow (Y_i = y) \sim \alpha(x)\ g(x,y)\ \lambda(dy) + (1 - \alpha(x))\ \ \delta_c(dy)$$

$$\oplus \quad \text{Clutter} \quad \mathcal{Y}' \quad \text{Poisson with intensity } \nu(dy) = h(y)\ \lambda(dy)$$

- Only observable $\mathcal{Y}^0 = \mathcal{Y} \times 1_{\neq c}$

$$\widehat{\gamma}(f) := \mathbb{E}\left(\mathcal{X}(f) \mid \mathcal{Y}^{\mathrm{o}}\right) = \gamma((1-\alpha)f) + \int \mathcal{Y}^{\mathrm{o}}(dy)\left(1 - \beta_\gamma(y)\right)\ \Psi_{\alpha g(y,\,.)}(\eta)(f)$$

with "the conditional clutter probability density"

$$\beta_\gamma(y) = h(y)/[h(y) + \gamma(\alpha g(y,\,.))]$$

Example full detection and no clutter $\alpha = 1$ & $h = 0$

$$\widehat{\gamma}(f) := \int \mathcal{Y}^{\mathrm{o}}(dy)\Psi_{g(y,\,.\,)}(\eta)(f)$$

$$\Downarrow$$

Conditional mean number of targets and "their conditional distributions"

$$\widehat{\gamma}(1) = \mathcal{Y}^{\mathrm{o}}(1) \quad \text{and} \quad \widehat{\eta}(f) := \frac{\widehat{\gamma}(f)}{\widehat{\gamma}(1)} = \int \frac{\mathcal{Y}^{\mathrm{o}}(dy)}{\mathcal{Y}^{\mathrm{o}}(1)} \underbrace{\Psi_{g(y,\,.\,)}(\eta)(f)}_{\text{Bayes' formula}}$$

PHD filtering equation
[with the prediction stage as the signal branching model (Q_n, μ_n)]

Hypothesis:

$\mathcal{X}_{n+1}$ **Poisson with intensity** $\gamma_{n+1} = \widehat{\gamma}_n Q_n + \mu_n \rightsquigarrow$ **with obs.** $\mathcal{Y}^0_{n+1}$ **as before**

$\Longrightarrow$ **PHD filtering equations:**

$$\begin{aligned} \gamma_{n+1} &:= \widehat{\gamma}_n Q_n + \mu_n \\ \widehat{\gamma}_n(f) &:= \gamma_n((1-\alpha_n)f) + \int \mathcal{Y}^o_n(dy)\,(1-\beta_{\gamma_n}(y))\ \Psi_{\alpha_n g_n(y,\,.\,)}(\gamma_n)(f) \end{aligned}$$

$\subset$ **A class of positive measure valued equations** $\supset$ Bernoulli filters, etc.

$$\gamma_{n+1}(dx') = \gamma_n Q_{n+1,\gamma_n}(dx') = \int \gamma_n(dx)\ Q_{n,\gamma_n}(x, dx')$$

with

$$Q_{n+1,\gamma_n}(x,dx') = Q_{n+1,\gamma_n}(1)(x) \times \frac{Q_{n+1,\gamma_n}(x,dx')}{Q_{n+1,\gamma_n}(1)(x)} = \underbrace{G_{n,\gamma_n}(x)}_{\text{potential function}} \times \underbrace{K_{n+1,\gamma_n}(x,dx')}_{\text{Markov transition}}$$

Stability properties of meas. valued equations

$$\eta_n = \gamma_n/\gamma_n(1) \rightsquigarrow \text{Nonlinear semigroup} \quad (\gamma_n(1),\eta_n) = \Gamma_{p,n}(\gamma_p(1),\eta_p)$$

- **Hypothesis on** $(m,\eta) \mapsto G_{n,m\eta}$ **s.t.** $m_n = \gamma_n(1) \in I_n = \text{compact} \subset]0,\infty[$
- **Stability Theorem for** $\Gamma_{p,n} = \left(\Gamma^1_{p,n}, \Gamma^2_{p,n}\right)$
  1. $\forall \eta_n$, the sg of total masses $\gamma_n(1)$ is expo. stable.
  2. $\forall m_n \in I_n$, the the sg of proba η_n is expo. stable.
  3. $m \in I_n \mapsto \Gamma^1_{n+1}(m,\eta_n)$ and $\eta \mapsto \Gamma^2_{n+1}(m_n,\eta)$ are Lipschitz continuous

  $$(3) \geq (1) \ \&\ (2) \ \Rightarrow \forall i = 1,2 \quad \left\|\Gamma^i_{p,n}(m',\eta') - \Gamma^i_{p,n}(m,\eta)\right\| \leq c\ e^{-\lambda(n-p)}$$
- Applications:

  3 natural conditions for the exponential stability of the PHD filter
  1. **small clutter intensities**
  2. **high detection probability**
  3. **high spontaneous birth rates**

Summary

Nonlinear equations

$$\gamma_{n+1} = \gamma_n Q_{n+1,\gamma_n} \quad \rightsquigarrow \quad \eta_n := \gamma_n/\gamma_n(1) \quad \text{and} \quad G_{n,\gamma_n} = Q_{n+1,\gamma_n}(1)$$

$$\Downarrow$$

- **The total mass evolution**

$$\gamma_{n+1}(1) = \gamma_n(1)\ \eta_n(G_{n,\gamma_n(1)\eta_n})$$

- **The "nonlinear filtering" conservative equations**

$$\eta_{n+1}(f) = \frac{\eta_n Q_{n,\gamma_n(1)\eta_n}(f)}{\eta_n Q_{n,\gamma_n(1)\eta_n}(1)} := \eta_n K_{n,\gamma_n(1)\eta_n}(f)$$

Mean field particle models

$$\eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_n^{(N,i)}} \simeq_{N\uparrow\infty} \eta_n \quad \text{and} \quad \gamma_n^N(1) \simeq_{N\uparrow\infty} \gamma_n(1)$$

with

$$\begin{array}{rcl} \gamma_{n+1}^N(1) & = & \gamma_n^N(1) \times \eta_n^N(G_{n,\gamma_n^N(1)\eta_n^N}) \\ \xi_{n+1}^{(N,i)} & = & \text{random var. with law} \quad K_{n+1,(\gamma_n^N(1),\eta_n^N)}(\xi_n^{(N,i)}, dx) \end{array}$$

$$\Downarrow$$

Same theorems as before with uniform convergence estimates

- $\supset$ $\forall$ scheme with local error controls
- $\supset$ Interacting Kalman type filters $\rightsquigarrow$ particle associations ($\simeq$ GM-PHD)
- $\supset$ Particle association $\oplus$ local branching models