Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes, No. 7

Consultations (RC 5112):

Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

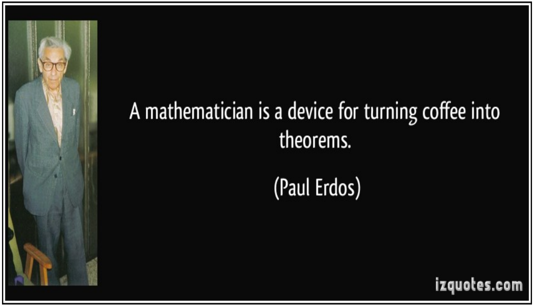
Reminder + Information

References in the slides

- **Material for research projects** $\rightsquigarrow$ Moodle
  (*Stochastic Processes and Applications* $\ni$ variety of applications)
- **Important results**

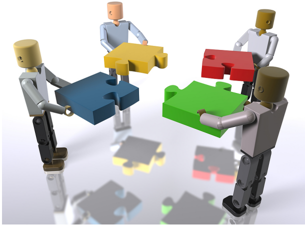
  $\subset$ **Assessment/Final exam** $=$ LOGO $=$

Citations of the day



– Paul Erdös (1913-1996)

Plan of the lecture



- Jump -diffusion processes
  - Time discretization / Simulation tool
  - Ito calculus
  - Infinitesimal generators
  - Martingale decompositions
  - Link to integro-differential equations
- Some applications:
  - Birth & death / Logistic processes
  - Moran genetic models $\oplus$ diffusion limit
  - Cellular dynamic models
- Nonlinear Markov models
  - Mean field interacting particle
  - Self-interacting - Reinforcement
  - Applications : fluid mechanics, computational physics,. . .

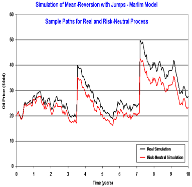
Three objectives

- **Recognize** a stochastic process given its (infinitesimal) generator
  - ⊕ perform Ito-Taylor type expansions
  - ⊕ **Inversely**, describe a process with its generator
  - ⊕ *DESIGN YOUR OWN STOCH PROCESSES!*
- **Simulate** **any** process using the discrete time interpretation.
- **When possible (quite rare)** work out explicit calculations.

Jump processes (simplify notation $d = 1$)

- **Between jump times T_n**

$$dX_t = b_t(X_t)\ dt\ +\ \sigma_t(X_t)\ dW_t \qquad T_n \leq t < T_{n+1}$$



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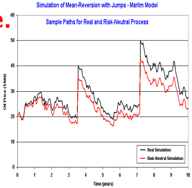

$$T_{n+1} = \inf \left\{ T_n \leq t\ :\ \int_{T_n}^{t} \lambda_s(X_s)\ ds\ \geq E_{n+1} \right\}$$

E_n are i.i.d. exponential r.v. with unit parameter.

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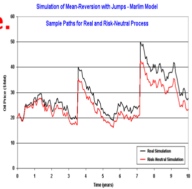

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- **Jump selection (Markov) transition $S_t(x, dy)$:**

$$X_{T_{n+1}-} \rightsquigarrow X_{T_{n+1}} \text{ r.v. with distribution } S_{T_{n+1}}(X_{T_{n+1}-}, dx)$$

Geo. clocks time mesh "dt"

Discrete integrals on the time mesh

$$T_{n+1} \quad = \quad \inf\left\{ T_n \leq t \;:\; e^{-\sum_{T_n \leq s < t} \;\lambda_s(X_s)\; ds} \leq e^{-E_{n+1}} \right\}$$

Geo. clocks time mesh "dt"

Discrete integrals on the time mesh

$$\begin{aligned} T_{n+1} &= \inf\left\{ T_n \leq t \; : \; e^{-\sum_{T_n \leq s < t} \; \lambda_s(X_s) \; ds} \leq e^{-E_{n+1}} \right\} \\ &= \inf\left\{ T_n \leq t \; : \; \prod_{T_n \leq s < t} e^{-\lambda_s(X_s) \; ds} \leq U_{n+1} \right\} \end{aligned}$$

with uniform i.i.d. r.v. U_n on $[0,1]$.

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$$\Downarrow$$

Geometric jump-times/clocks on the time mesh

$$\mathbb{P}\left(T_{n+1} \in [\mathbf{t}, \mathbf{t} + \mathbf{dt}] \mid X_s,\; s \leq \mathbf{t}\right)$$

$$= \left(\prod_{T_n \leq s < \mathbf{t}} e^{-\lambda_s(X_s) ds}\right) \left(1 - e^{-\lambda_{\mathbf{t}}(X_{\mathbf{t}})\mathbf{dt}}\right)$$

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$$\Downarrow$$

Jump time T after "t"

$$\mathbb{P}\left(T = t+dt\ , X_{t+dt} \in dx \mid X_t\right) = \lambda_t(X_t)\ dt\ S_t(X_t, dx)$$

Bernoulli model on the time mesh "dt"

Given X_t, *description of the increment* $\Delta X_t = X_{t+dt} - X_t$

$$\mathbf{Y_t} = X_t + b_t(X_t)\ dt\ +\ \sigma_t(X_t)\ (W_{t+dt} - W_t)$$

$$\mathbb{P}\left(X_{t+dt} \in dx \mid \mathbf{Y_t}\right) = e^{-\lambda_t(\mathbf{Y_t})dt}\ \delta_{\mathbf{Y_t}}(dx) + \left(1 - e^{-\lambda_t(\mathbf{Y_t})dt}\right)\ S_t(\mathbf{Y_t}, dx)$$

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$$\Updownarrow$$

Bernoulli process

$$X_{t+dt} = (1-\epsilon_t)\ \mathbf{Y_t}\ +\ \epsilon_t\ Z_t \quad \text{with} \quad Z_t\ \sim\ S_t(\mathbf{Y_t}, dx)$$

and the $\{0,1\}$*-valued r.v.* ϵ_t *with jump probability*

$$\mathbb{P}\left(\epsilon_t = 1 \mid \mathbf{Y_t}\right) = 1 - e^{-\lambda_t(\mathbf{Y_t})dt} \simeq \lambda_t(\mathbf{Y_t})dt$$

Related Stoch. models

- $\sigma_t = 0 \rightsquigarrow$ Piecewise deterministic Markov processes (PDMP).

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At jump times (exponential inter-times with unit parameter):

$$X_0 = \mathcal{X}_0 \quad \rightsquigarrow \quad X_{T_1} = \mathcal{X}_1 \quad \rightsquigarrow \quad X_{T_2} = \mathcal{X}_2 \ldots \quad \rightsquigarrow \quad X_{T_n} = \mathcal{X}_n$$

Also called *Pure jump processes*.

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- $\lambda_t = 0 \rightsquigarrow$ Stochastic Diffusion equation (SDE).
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  - $\lambda_t(x) = \lambda \rightsquigarrow$ Poisson process $X_t = N_t$ with intensity λ.
  - $\lambda_t(x) = \lambda_t \rightsquigarrow$ Non-homogeneous Poisson process $\sim \lambda_t$.

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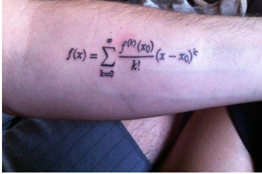
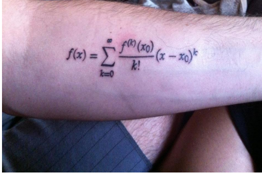
- *General case* $\rightsquigarrow$ (Marked)-Jump-diffusion models.

Doeblin-Ito-Taylor expansion



$$dX_t = dX_t^c + \Delta X_t$$

Doeblin-Ito-Taylor expansion ⚠



$$dX_t = dX_t^c + \Delta X_t = dX_t^c \quad \textbf{or} \quad \Delta X_t$$

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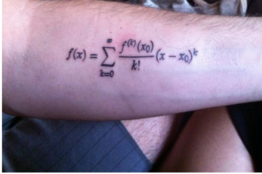

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$$\Downarrow$$

$$df(t, X_t)$$

$$= f(t + dt, X_t + dX_t) - f(t, X_t)$$

$$= \underbrace{\frac{\partial f}{\partial t}(t, X_t)dt \; + \; f'(X_t)\; dX_t^c + \frac{1}{2}\; f''(X_t)\; dX_t^c dX_t^c}_{=\left[\frac{\partial}{\partial t} + L_t^c\right](f)(t,X_t)\; dt \; + \; dM_t^c(f)} + \Delta f(t, X_t)$$

with the infinitesimal generator

$$L_t^c = b_t \; \frac{\partial}{\partial x} + \frac{1}{2}\; \sigma_t^2 \; \frac{\partial^2}{\partial x^2}$$

and the martingale increment

$$dM_t^c(f) = f'(X_t)\; \sigma_t(X_t)\; dW_t$$

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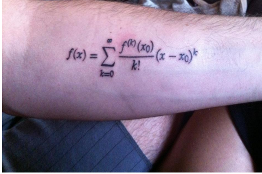

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$$dM_t^c(f) = f'(X_t)\; \sigma_t(X_t)\; dW_t \qquad \textbf{AND} \qquad \boldsymbol{\Delta f(t, X_t)} \quad \textbf{???}$$

The jump generator

$$\mathbb{P}\left(T = t + dt\ , X_{t+dt} \in dx \mid X_t\right) = \lambda_t(X_t)\ dt\ S_t(X_t, dx)$$

$$\Downarrow$$

$$\begin{aligned}\mathbb{E}\left(\Delta f(t, X_t) \mid X_t = x\right) &= \lambda_t(x)\ dt \int \left(f(t,y) - f(t,x)\right)\ S_t(x, dy) \\ &:= L_t^d(f)(x)\ dt\end{aligned}$$

$$\Downarrow$$

Predictable and martingale parts

$$\begin{aligned}\Delta f(t, X_t) &= \mathbb{E}\left(\Delta f(t, X_t) \mid \mathcal{F}_t\right) + \Delta f(t, X_t) - \mathbb{E}\left(\Delta f(t, X_t) \mid \mathcal{F}_t\right) \\ &= L_t^d(f)(X_t)\ dt + dM_t^d(f)\end{aligned}$$

The angle brackets

$$dM_t^d(f) = \Delta f(t, X_t) - \overbrace{\mathbb{E}\left(\Delta f(t, X_t) \mid \mathcal{F}_t\right)}^{=\ldots dt}$$

$$\Downarrow$$

$$\begin{aligned}
\mathbb{E}\left((dM_t^d(f))^2 \mid \mathcal{F}_t\right) &= \mathbb{E}\left((\Delta f(t, X_t))^2 \mid \mathcal{F}_t\right) \\
&= \lambda_t(X_t)\ dt \int \ (f(t, y) - f(t, X_t))^2\ S_t(X_t, dy) \\
&= L_t^d[(f - f(x))^2](x)_{|\ x=X_t}\ dt
\end{aligned}$$

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&= L_t^d[(f - f(x))^2](x)_{|\ x = X_t}\ dt = \Gamma_{L_t^d}(f, f)(X_t)\ dt
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$$\Downarrow$$

The angle bracket of the martingale $M_t^d(f)$

$$\langle M^d(f)\rangle_t \; = \; \int_0^t \Gamma_{L_s^d}(f, f)(X_s)\ ds$$

Finally. . . the general rule ⚠

$$
\begin{aligned}
& df(t, X_t) \\
&= \left[\tfrac{\partial}{\partial t} + L_t^c\right](f)(t, X_t)\, dt \;+\; dM_t^c(f) \;+\; L_t^c(f)(t, X_t)\, dt + dM^d(f) \\
&= \left[\tfrac{\partial}{\partial t} + L_t\right](f)(t, X_t)\, dt \;+\; dM_t(f)
\end{aligned}
$$

with

$$L_t = L_t^c + L_t^d \quad \text{and} \quad M_t(f) = M_t^c(f) + M_t^d(f)$$

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$$L_t = L_t^c + L_t^d \quad \text{and} \quad M_t(f) = M_t^c(f) + M_t^d(f)$$

and a martingale $M_t(f)$ with angle bracket

$$\frac{d}{dt}\,\langle M^d(f)\rangle_t \;= \Gamma_{L_t^c}(f, f)(X_t) \;+\; \Gamma_{L_t^d}(f, f)(X_t) = \Gamma_{L_t}(f, f)(X_t)$$

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Notice

$$\begin{aligned}
\Gamma_{L_t^d+L_t^d}(f,f)(x) &= (L_t^c+L_t^d)[(f-f(x))^2](x)\\
&= L_t^c[(f-f(x))^2](x) + L_t^d[(f-f(x))^2](x)
\end{aligned}$$

An example (quite rare but important)

$$L_t^d(f)(x) = \underbrace{\lambda_t(x)}_{:=1(\rightsquigarrow \text{Poisson proc.})} \int (f(y) - f(x)) \underbrace{S_t(x, dy)}_{=\delta_{x+c_t x}(dy)}$$

$$L_t^c(f)(x) = b_t x \; f'(x) + \frac{1}{2} (\sigma_t x)^2 \; f''(x)$$

$$\Downarrow$$

$$dX_t = \underbrace{b_t \; X_t \; dt + \; \sigma_t \; X_t \; dW_t}_{=dX_t^c} \; + \; \underbrace{c_t \; X_{t-} \; dN_t}_{=\Delta X_t}$$

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$$\Downarrow$$

$$d \log X_t = b_t \; dt + \; \sigma_t \; dW_t - \frac{1}{2} \; \sigma_t^2 \; dt + \log(1 + c_t) \; dN_t$$

$$\Downarrow$$

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$$dX_t = \underbrace{b_t \; X_t \; dt + \; \sigma_t \; X_t \; dW_t}_{=dX_t^c} \; + \; \underbrace{c_t \; X_{t-} \; dN_t}_{=\Delta X_t}$$

$$\Downarrow$$

$$d \log X_t = b_t \; dt + \; \sigma_t \; dW_t - \frac{1}{2} \; \sigma_t^2 \; dt + \log\left(1 + c_t\right) \; dN_t$$

$$\Downarrow$$

Geometric Brownian-Poisson process

$$X_t = X_0 \; \exp\left(\int_0^t \left(b_s - \frac{1}{2} \; \sigma_s^2 \right) \; ds + \int_0^t \sigma_s \; dW_s + \int_0^t \log\left(1 + c_s\right) \; dN_s \right)$$

Integro-diff. eq. $S_t(x, dy) = s_t(x, y)\ dy$

$$df(X_t) = (L_t^c + L_t^d)(f)(X_t)\ dt\ +\ \text{Martingale increment}$$

$$\Updownarrow \quad \boldsymbol{[p_t(x)dx = \mathbb{P}(X_t \in dx)]} \textbf{ whenever it exists}$$

$$d\mathbb{E}\left(f(X_t)\right) \;=\; \int L_t^c(f)(x)\ p_t(x)\ dx + \int L_t^d(f)(x)\ p_t(x)\ dx$$

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Using the fact that

$$\int\ \lambda_t(x)\ \int (f(y) - f(x))\ s_t(x, y)\ p_t(x)\ dx\ dy$$

$$= \int f(x)\ \left[\int\ \lambda_t(y)\ p_t(y)\ s_t(y, x)\ dy - \lambda_t(x)\ p_t(x)\right]\ dx$$

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we prove

$$\frac{\partial p_t}{\partial t} = -\frac{\partial(b_t p_t)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma_t^2 p_t)}{\partial x^2} + \int\ \lambda_t(y)\ p_t(y)\ S_t(y, dx)\ \ dy - \lambda_t(x)\ p_t(x)$$

Logistic proc./birth and death process

Epidemic evolution - State $\{0, 1, \ldots, d\}$

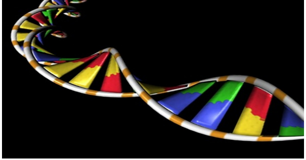
$$L(f)(x) = \lambda_{infect}(x)\ [f(x+1) - f(x)] + \lambda_{recover}(x)\ [f(x-1) - f(x)]$$

with the infection and the recovery rates

$$\lambda_{infect}(x) = \alpha\ d^2\ \frac{x}{d}\left(1 - \frac{x}{d}\right) \quad \text{and} \quad \lambda_{recover}(x) = \beta\ d\ \frac{x}{d}$$

- $\frac{x}{d}$ = proportion of infected individuals.
- Every infected individual (in a pool of x) remains infected for an exponential rate β.

The Moran genetic model



State $\{0/d, 1/d, \ldots, d/d\}$=% alleles of type A in a genetic model with two alleles A and B.

$$L(f)(x) = \binom{d}{2} \, x(1-x) \, [f(x+1/d) - f(x)]$$

$$+ \binom{d}{2} \, x(1-x) \, [f(x-1/d) - f(x)]$$

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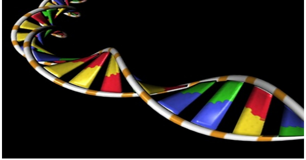

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$$+ \binom{d}{2} \; x(1-x) \; [f(x-1/d) - f(x)]$$

Diffusion limits

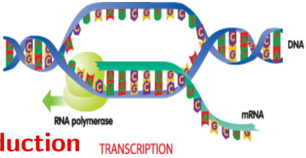
$$f(x+1/d) - f(x) = f'(x)/d + \frac{1}{2} \; f''(x)/d^2 + \mathrm{O}(1/d^3)$$

This implies that

$$L(f)(x) = \frac{d(d-1)}{2} \; x(1-x) \; \left([f(x+1/d) - f(x)] + [f(x-1/d) - f(x)]\right)$$

$$= \frac{1}{2} \; x(1-x) \; f''(x) + \mathrm{O}(1/d)$$

Cellular dynamic models



Genes (DNA)
$\rightsquigarrow$ **Messenger molecule (mRNA)** $\rightsquigarrow$ **protein production**

$$
\begin{array}{rcll}
\text{Gene} & \xrightarrow{\lambda_1} & \text{Gene} + \text{mRNA} & \textit{(production of mRNA)} \\
\text{mRNA} & \xrightarrow{\lambda_2} & \text{mRNA} + \text{Proteins} & \textit{(production of Proteins)} \\
\text{mRNA} & \xrightarrow{\lambda_3} & \emptyset \quad \& \quad \text{Proteins} \xrightarrow{\lambda_4} \emptyset & \textit{(degradation)}
\end{array}
$$

$$\Updownarrow$$

Jump process:

$$X_t = (Y_t, Z_t) = (\text{number of mRNA-Molecules, Proteins}).$$

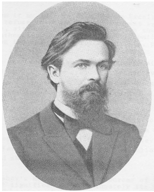
$$
\begin{aligned}
L(f)(y,z) \quad = \quad & \lambda_1\,[f(y+1,z) - f(y,z)] + \lambda_2 y\,[f(y,z+1) - f(y,z)] \\
& + \lambda_3 y\,[f(y-1,z) - f(y,z)] + \lambda_4 z\,[f(y,z-1) - f(y,z)]
\end{aligned}
$$

BREAK

Nonlinear stochastic processes

- Nonlinear Markov chains $\rightsquigarrow$ **interacting** particle systems.
- Self-interacting Markov chains $\rightsquigarrow$ **reinforcement** process.
- Discrete or Continuous time models

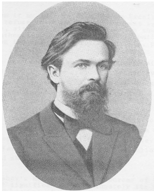
Markov chains



(Elementary) Markov transitions:

$$\mathbb{P}\left(X_n \in dx_n \mid X_{n-1} = x_{n-1}\right) = M_n(x_{n-1}, dx_n)$$
$$\stackrel{ex.1}{=} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(x_n - a_n(x_{n-1})\right)\right\} dx_n$$

Markov chains



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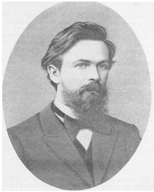
- **Linear evolution model**

$$\eta_n = \mathrm{Law}(X_n) \rightsquigarrow \underbrace{\eta_n = \Phi_n(\eta_{n-1}) := \eta_{n-1} M_n}$$

Markov chains



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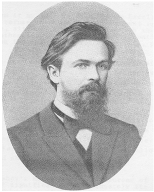
Markov chains



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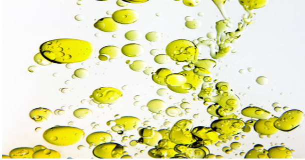
- **Simulation:** Nice and usually easy

$$\text{Ex.1} \rightsquigarrow X_n = a_n(X_{n-1}) + W_n \quad \text{with} \quad W_n \sim N(0,1)$$

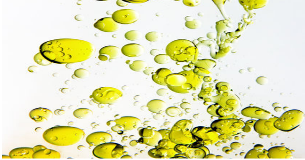
Nonlinear Markov chains



Transitions $\sim$ the law of the random states

$$\mathbb{P}\left(X_n \in dx_n \mid X_{n-1}\right) = M_{n,\eta_{n-1}}(X_{n-1}, dx_n) \quad \text{with} \quad \eta_{n-1} = \mathrm{Law}(X_{n-1})$$

Nonlinear Markov chains



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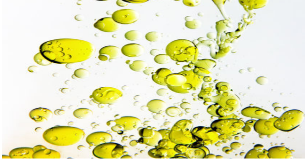
Nonlinear Markov chains



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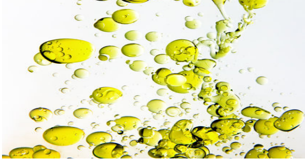
Example:

$$\begin{aligned} X_n &= \int a_n(X_{n-1}, y)\; \eta_{n-1}(dy) + W_n \\ &= \int a_n(X_{n-1}, y)\; \mathbb{P}(X_{n-1} \in dy) + W_n \end{aligned}$$

Nonlinear Markov chains



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$$\Downarrow \quad \boldsymbol{a_n(x,y) = b_n(x) + c_n(y)}$$

$$X_n \quad = \quad b_n(X_{n-1}) + \mathbb{E}\left(c_n(X_{n-1})\right) + W_n \quad \textbf{Simulation???}$$

An example ($n = 1$)

$$\forall 1 \leq i \leq N \qquad X_1^i = X_0^i + \mathbb{E}\left(\frac{\sin(2\pi\lfloor X_0 \rfloor)}{1 + X_0^2}\right) + W_1^i$$ **Simulation???**

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Reminder - The law of large numbers:

$$\int f(x_0)\,\eta_0(dx_0) := \mathbb{E}(f(X_0)) \simeq \frac{1}{N}\sum_{1\leq i\leq N} f(X_0^i) := \int f(x_0) \underbrace{\eta_0^N(dx_0)}_{:=\frac{1}{N}\sum_{1\leq i\leq N}\delta_{X_0^i}}$$

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$$\Downarrow$$

Simulation of the first transition

$$\forall 1 \leq i \leq N \qquad X_1^i \simeq X_0^i + \frac{1}{N}\sum_{1\leq j\leq N}\frac{\sin(2\pi\lfloor X_0^j \rfloor)}{1 + (X_0^j)^2} + W_1^i$$

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$$\Downarrow$$

$(X_1^1, \ldots, X_1^N)$ almost i.i.d. copies of X_1

An example ($n = 2$)

$$X_2^i = X_1^i + \mathbb{E}\left(\frac{\sin(2\pi\lfloor X_1 \rfloor)}{1 + X_1^2}\right) + W_2^i$$ **Simulation???**

An example ($n = 2$)

$$X_2^i = X_1^i + \mathbb{E}\left(\frac{\sin(2\pi\lfloor X_1\rfloor)}{1+X_1^2}\right) + W_2^i$$ **Simulation???**

$$\Downarrow$$

$$X_2^i \simeq X_1^i + \frac{1}{N}\sum_{1\leq j\leq N}\frac{\sin(2\pi\lfloor X_1^j\rfloor)}{1+(X_1^j)^2} + W_2^i$$

$$\Downarrow$$

$(X_2^1, \ldots, X_2^N)$ almost i.i.d. copies of X_2 $\quad \ldots/\ldots$

Illustrations

- McKean-Vlasov fluid diffusions models
- Fouque-Sun Financial systemic risk model
- Dyson model - Gaussian Orthogonal Ensemble
- Feynman-Kac models - Interacting jumps.

$$\Downarrow$$

$\rightsquigarrow$ scilab prog. $+$ more info (Tutorial X 3rd year) [solutions on demand]

An example ($n = 1$)

"Diffusion like model"

$$
\begin{aligned}
\Delta X_n \quad &:= \quad b_n(X_{n-1}, \eta_{n-1}) \quad + \quad \sigma_n(X_{n-1}, \eta_{n-1}) \quad W_n \\
&\overset{ex.}{=} \left[\int \alpha_n(X_{n-1}, x)\ \eta_{n-1}(dx)\right] + \left[\int \beta_n(X_{n-1}, x)\eta_{n-1}(dx)\right]\ W_n
\end{aligned}
$$

$$\Downarrow$$

Mean field particle model (N particles $i = 1, \ldots, N$)

$$
\begin{aligned}
\Delta X_n^i \quad &:= \quad b_n(X_{n-1}^i, \eta_{n-1}^N) + \sigma_n(X_{n-1}^i, \eta_{n-1}^N)\ W_n^i \\
&\overset{ex.}{=} \quad \frac{1}{N} \sum_{1 \leq j \leq N} \alpha_n(X_{n-1}^i, X_{n-1}^j) + \frac{1}{N} \sum_{1 \leq j \leq N} \beta_n(X_{n-1}^i, X_{n-1}^j)\ W_n^i
\end{aligned}
$$

Application domains:

Condense matter models, fluid mechanics, McKean-Vlasov diffusions, . . .

Fouque-Sun Financial systemic risk mode

$\subset$ J.P. Fouque & L. H. Sun Systemic Risk (math finance)

X_t^i = **log-monetary reserve of banks**

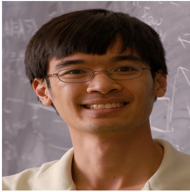
- Borrowing and lending to each other
- Mean reversion rate α
- Interaction degree [swarming behavior]

$$\Downarrow$$

Stochastic (toy) model:

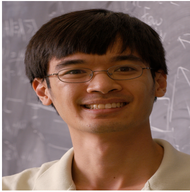
$$dX_t^i = \underbrace{\alpha \frac{1}{N} \sum_{1 \le j \le N} (X_t^j - X_t^i)}_{\text{Attraction}} \, dt + \sigma \, dW_t^i$$

Dyson model - Gaussian Orthogonal Ensemble



$\subset$ Terry Tao Blog Notes
(Field Medal 06 - Adelaide - Australia (HK origin), IQ 230)

Dyson model - Gaussian Orthogonal Ensemble



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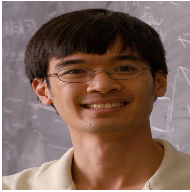
Dyson equations:

$$d\lambda_t^i = \underbrace{\frac{1}{N} \sum_{1 \leq j \neq i \leq N} \frac{1}{(\lambda_t^i - \lambda_t^j)}}_{\text{Repulsion}} \, dt \; + \; \frac{2}{N} \, dW_t^i$$

Dyson model - Gaussian Orthogonal Ensemble



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$$\Updownarrow$$

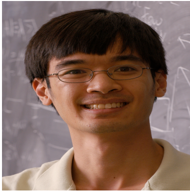
Gaussian Orthogonal Ensemble

$$(\lambda_t^1 < \ldots < \lambda_t^N) \;\; = \;\; \text{Eigenvalues of a symmetric random matrix } A(t)$$

$$A_{i,i}(t) = \sqrt{\frac{2}{N}} \; W_t^i \quad \text{and} \quad A_{i,j}(t) = A_{j,i}(t) = \sqrt{\frac{1}{N}} \; W_t^{i,j}$$

with $N(N-1)/2$ i.i.d. Brownian motions.

Dyson model - Gaussian Orthogonal Ensemble



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$$\Updownarrow$$

N Brownian motions | no $\cap$ (N large)Tut. 3rd year X (french)

Jump type models

$$\eta_n = \Phi_n(\eta_{n-1}) = \eta_{n-1} M_{n,\eta_{n-1}}$$

Acceptance/Rejection type transitions

$$M_{n,\eta_{n-1}}(x, dy)$$

$$= \mathbf{G(x)}\ K(x, dy)\ +\ (\mathbf{1 - G(x)})\ \frac{1}{\eta_{n-1}(G)}\ \int\ \eta_{n-1}(dz)\ G(z)\ K(z, dy)$$

associated with some $G(x) \in [0, 1]$ and some Markov transition $K(x, dy)$.

- $G(x) = 1 \Longrightarrow M_{n,\eta_{n-1}} = K \rightsquigarrow$ Markov chain sampling
- $K(x, dy) = \delta_x(dy)$:

$$\Rightarrow M_{n,\eta_{n-1}}(x, dy)$$

$$= \mathbf{G(x)}\ \delta_x(dy) + (\mathbf{1 - G(x)})\ \frac{1}{\eta_{n-1}(G)}\ \eta_{n-1}(dz)\ G(z)\ \delta_z(dy)$$

Jump type models

Mean field particle model

$$(X^1_{n-1}, \ldots, X^N_{n-1}) \rightsquigarrow (X^1_n, \ldots, X^N_n)$$

with

$$\forall 1 \leq i \leq N \qquad X^i_n \sim M_{n,\eta^N_{n-1}}(X^i_{n-1}, dy)$$

$$\Downarrow$$

Jump type transitions

$$\begin{aligned} M_{n,\eta^N_{n-1}}(X^i_{n-1}, dy) &= \mathbf{G(X^i_{n-1})}\, K(X^i_{n-1}, dy) \\ &\quad + (\mathbf{1} - \mathbf{G(X^i_{n-1})}) \sum_{1 \leq k \leq N} \frac{G(X^k_{n-1})}{\sum_{1 \leq j \leq N} G(X^j_{n-1})}\, K(X^k_{n-1}, dy) \end{aligned}$$

$\subset$ **Feynman-Kac models (cf. section 8.4 & 8.5)** $\rightsquigarrow$ **Application:**
Any conditional distribution, molecular chemistry, interacting MCMC, . . .

Self-interacting chains/Reinforced processes

Elementary transitions

$$\mathbb{P}\left(X_n \in dx_n \mid X_0, \ldots, X_{n-1}\right) = M_{n,\eta^{n-1}}(X_{n-1}, dx_n)$$

depending on the empirical measure of the historical process

$$\eta^{n-1} = \frac{1}{n} \sum_{0 \leq p < n} \delta_{X_p}$$

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Example 1:

$$\begin{aligned} X_n &= \int a_n(X_{n-1}, y)\ \eta^{n-1}(dy) + W_n \\ &= \frac{1}{n} \sum_{0 \leq p < n} a_n(X_{n-1}, X_p)\ + W_n \end{aligned}$$

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$$\Downarrow \quad a_n(x,y) = b_n(x) + c_n(y)$$

$$X_n = b_n(X_{n-1}) + \frac{1}{n} \sum_{0 \leq p < n} c_n(X_p) + W_n$$

Other illustrations

Example 1: Given some probability measure μ

$$M_{n,\eta^{n-1}}(X_{n-1}, dx_n) = \epsilon\ \eta^{n-1}(dx_n) + (1-\epsilon)\ \mu(dx_n)$$

Other illustrations

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Example 2: $\rightsquigarrow \rightsquigarrow$ *Acceptance/Rejection type transitions*

$$M_{n,\eta^{n-1}}(x, dy)$$

$$= \mathbf{G}(\mathbf{x})\ K(x, dy)\ +\ (\mathbf{1} - \mathbf{G}(\mathbf{x}))\ \frac{1}{\eta^{n-1}(G)}\ \int\ \eta^{n-1}(dz)\ G(z)\ K(z, dy)$$

associated with some $G(x) \in [0,1]$ and some Markov transition $K(x, dy)$.

Other illustrations

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$\ldots / \ldots$