Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes, No. 6

Consultations (RC 5112):

Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

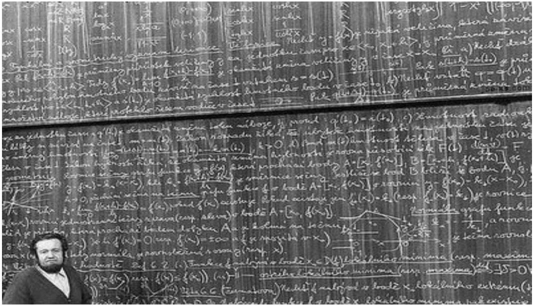
Reminder + Information

References in the slides

- **Material for research projects** $\rightsquigarrow$ Moodle
  (*Stochastic Processes and Applications* $\ni$ variety of applications)
- **Important results**

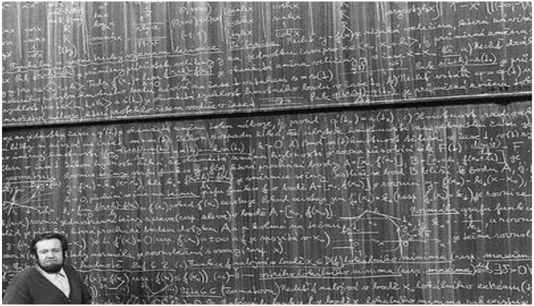
  $\subset$ **Assessment/Final exam** $=$ LOGO $=$

Citations of the day



The essence of mathematics is not to make simple things complicated, but to make complicated things simple. S. Gudder

Citations of the day



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2 citations - US National Academy of Science states

NATIONAL ACADEMY OF SCIENCES 1863–2013 Celebrating 150 Years of Service to the Nation

If one disqualifies the Pythagorean Theorem from contention, it is hard to think of a mathematical result which is better known and more widely applied in the world today than **"Itō's Lemma"**.

This result holds the same position in stochastic analysis that Newton's fundamental theorem holds in classical analysis. That is, it is the sine qua non of the subject.

$$\Downarrow$$

Everything related to "this lemma" = Itō-Doeblin formula is

....

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Plan of the lecture

- Reminder + multi-dimensional Brownian motions
- Diffusion processes:
  - Ito Calculus
  - Infinitesimal generators
  - Evolution semigroups
  - Fokker Planck equation
- Some applications:
  - Ornstein-Ulhenbeck processe
  - Geometric Brownian motion
  - Mathematical finance (Black-Scholes-Merton formula)

Reminder - Brownian motion W_t

Simple random walk model

$$\begin{aligned} W_{t+dt} := W_t + \epsilon_t\,\sqrt{dt} \quad &\text{with} \quad \epsilon_t := \pm 1 \ \ \text{Proba} \ \ 1/2 \\ &\underline{\textbf{or}} \quad\ \ \epsilon_t \ \sim\ \mathbf{N(0,1)} \end{aligned} \tag{1}$$

Reminder - Brownian motion W_t

Simple random walk model

$$W_{t+dt} := W_t + \epsilon_t \sqrt{dt} \quad \text{with} \quad \epsilon_t := \pm 1 \ \text{ Proba } \ 1/2$$
$$\underline{\textbf{or}} \qquad \epsilon_t \sim \mathbf{N(0,1)} \tag{1}$$

$\rightsquigarrow$ **Only "3 simple ingredients":**

$$(1) \Rightarrow \begin{cases} dW_t \times dW_t & = \overbrace{\epsilon_t^2}^{\mathbb{E}(\ldots)=1} \ dt \ \text{"} = \text{"} \ dt \\ dt \times dt & = 0 \\ dt \times dW_t & = dt \ \times \pm\sqrt{dt} = 0 \end{cases}$$

$\oplus$ **Randomness encapsulated in** $\mathcal{F}_t = \sigma(W_s \ : \ s \leq t)$ $\bigstar$.

2d-Brownian motion (abbreviated 2d-BM)

$W_t = (W_t^{(1)}, W_t^{(2)}) \sim$ **2 independent** $(\epsilon_t^{(1)}, \epsilon_t^{(2)})$

$$\Downarrow$$

$$\forall i, j \in \{1, 2\} \qquad W_{t+dt}^{(i)} - W_t^{(i)} = \epsilon_t^{(i)} \sqrt{dt} \tag{2}$$

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$$\Downarrow$$

$$(W_{t+dt}^{(1)} - W_t^{(1)}) \times (W_{t+dt}^{(2)} - W_t^{(2)}) = \underbrace{\epsilon_t^{(1)} \epsilon_t^{(2)}}_{\mathbb{E}(\dots)=0} dt$$

$$\Downarrow$$

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$$(W_{t+dt}^{(1)} - W_t^{(1)}) \times (W_{t+dt}^{(2)} - W_t^{(2)}) \;=\; \underbrace{\epsilon_t^{(1)}\epsilon_t^{(2)}}_{\mathbb{E}(\ldots)=0}\; dt$$

$$\Downarrow$$

$\rightsquigarrow$ **New ingredient:**

$$(2) \Rightarrow \forall i,j \in \{1,2\} \qquad dW_t^{(i)} \times dW_t^{(j)} = 1_{i=j}\; dt$$

$\oplus$ **Randomness encapsulated in** $\mathcal{F}_t = \sigma(W_s \;:\; s \le t)$ $\bigstar$.

[YouTube video]

2d- BM $W_t = (W_t^1, W_t^2)$

$$df(W_t) = f(W_t + dW_t) - f(W_t)$$

$$= \tfrac{\partial f}{\partial x^1}(W_t)\ dW_t^{(1)} + \tfrac{\partial f}{\partial x^2}(W_t)\ dW_t^{(2)}$$

$$+\tfrac{1}{2}\ \tfrac{\partial^2 f}{\partial x^1}(W_t)\ \underbrace{dW_t^{(1)} dW_t^{(1)}}_{=dt} + \tfrac{1}{2}\ \tfrac{\partial^2 f}{\partial x^2}(W_t)\ \underbrace{dW_t^{(2)} dW_t^{(2)}}_{=dt} + \text{"}O(dt\sqrt{dt})\text{"}$$

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$$\Downarrow$$

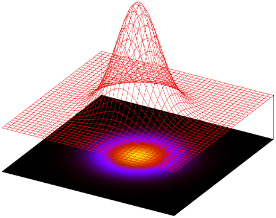
2d - Ito-Doeblin formula

$$df(W_t) = L(f)(W_t)\ dt + \underbrace{dM_t(f)}_{=dM_t^{(1)}(f)+dM_t^{(2)}(f)}$$

with

$$\begin{aligned} L(f) &= \Delta(f) \\ &:= \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} \qquad \& \qquad dM_t^{(i)}(f) = \frac{\partial f}{\partial x_i}(W_t)\ dW_t^{(i)} \end{aligned}$$

2d- BM & Heat eq.

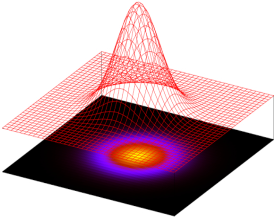


$$\mathbb{P}\left(W_t^{(1)} \in dx,\ W_t^{(2)} \in dy\right) = p_t(x,y)\ dxdy$$

$\Downarrow$ [Ito-Doeblin formula]

$$d\mathbb{E}\left(f(W_t)\right) = \int f(x,y)\ \frac{\partial}{\partial t} p_t(x,y)\ dxdy\ dt$$

2d- BM & Heat eq.

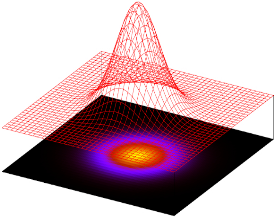


$$\mathbb{P}\left(W_t^{(1)} \in dx,\ W_t^{(2)} \in dy\right) = p_t(x,y)\ dxdy$$

$\Downarrow$ [Ito-Doeblin formula]

$$\begin{aligned} d\mathbb{E}\left(f(W_t)\right) &= \int f(x,y)\ \frac{\partial}{\partial t} p_t(x,y)\ dxdy\ dt \\ &= \int \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] f(x,y)\ p_t(x,y)\ dxdy\ dt \end{aligned}$$

2d- BM & Heat eq.



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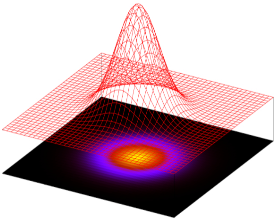
2d- BM & Heat eq.



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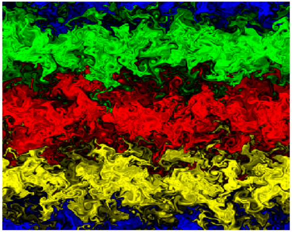
$$\Downarrow$$

2d - Heat equation

$$\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\ p_t(x,y)$$

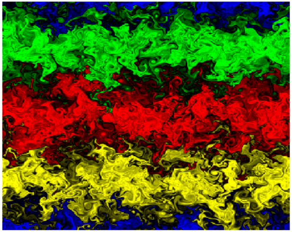
1d - diffusion processes



Simple Markov chain model on $\mathbb{R}$ (b_t, σ_t smooth + bounded)

$$X_{t+dt} - X_t := b_t(X_t)\ dt\ +\ \sigma_t(X_t)\ (W_{t+dt} - W_t)$$

1d - diffusion processes ⚠



Simple Markov chain model on $\mathbb{R}$ (b_t, σ_t smooth + bounded)

$$X_{t+dt} - X_t := b_t(X_t)\ dt\ +\ \sigma_t(X_t)\ (W_{t+dt} - W_t)$$

$$\Downarrow$$

One dimensional diffusion model

$$dX_t = b_t(X_t)dt + \sigma_t(X_t)dW_t$$

$\oplus$ **Randomness** $\oplus$ **information encapsulated in**

$$\mathcal{F}_t = \sigma(W_s\ :\ s \leq t) = \sigma(X_s,\ s \leq t)\quad \bigstar\bigstar$$

.

Ex.: Brownian fluid flow models

Fluid particle ($X_0 = 0$):

$$dX_t = [\ v\ +\ U'(X_t)]\ dt\ +\ \sqrt{2D}\ dW_t$$

- Fluid velocity flow v.
- Diffusion coefficient = D.
- Energy well = $U(x)$

  Example $U(x) = k\ x^2/2$; k = spring constant

$\rightsquigarrow$ *Wolfram -[Brownian-Fluid-model-(v,D).cdf*

$\oplus$ *Brownian-Motion-2D-TheFokker-Planck-Equation.cdf]*

1d - diffusion processes - Ito formula

$$dX_t = \underbrace{\mathbf{b_t(X_t)\ dt}}_{\text{drift term}} + \underbrace{\sigma_{\mathbf{t}}(\mathbf{X_t})\mathbf{dW_t}}_{\text{diffusion term}}$$

1d - diffusion processes - Ito formula

$$dX_t = \underbrace{\mathbf{b_t(X_t)\ dt}}_{\text{drift term}} + \underbrace{\sigma_{\mathbf{t}}\mathbf{(X_t)dW_t}}_{\text{diffusion term}}$$

$$\Downarrow$$

$$\begin{aligned} df(t, X_t) &= f(t + dt, X_t + dX_t) - f(t, X_t) \\ &= \frac{\partial f}{\partial t}(t, X_t)\ dt + \frac{\partial f}{\partial x}(t, X_t)\ dX_t + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}(t, X_t)\ dX_t\ dX_t \end{aligned}$$

1d - diffusion processes - Ito formula ⚠

$$dX_t = \underbrace{\mathbf{b_t(X_t)\ dt}}_{\text{drift term}} + \underbrace{\boldsymbol{\sigma}_{\mathbf{t}}\mathbf{(X_t)dW_t}}_{\text{diffusion term}}$$

$$\Downarrow$$

$$\begin{aligned}
df(t, X_t) &= f(t + dt, X_t + dX_t) - f(t, X_t) \\
&= \frac{\partial f}{\partial t}(t, X_t)\ dt + \frac{\partial f}{\partial x}(t, X_t)\ dX_t + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}(t, X_t)\ dX_t\ dX_t \\
&= \underbrace{\left[\frac{\partial f}{\partial t}(t, X_t)\ +\ \mathbf{b_t(X_t)}\ \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{t}, \mathbf{X_t}) + \frac{\mathbf{1}}{\mathbf{2}}\ \boldsymbol{\sigma}^{\mathbf{2}}_{\mathbf{t}}(\mathbf{X_t})\ \frac{\partial^{\mathbf{2}} \mathbf{f}}{\partial \mathbf{x}^{\mathbf{2}}}(\mathbf{t}, \mathbf{X_t})\right]}\ dt + \underbrace{\frac{\partial f}{\partial x}(t, X_t)\ \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{X_t})\ \mathbf{dW_t}} \\
&= \left(\frac{\partial}{\partial t} + L_t\right)(f)(t, X_t) \qquad dt + \qquad dM_t(f)
\end{aligned}$$

First key observation

$$\begin{aligned} df(t, X_t) &= f(t + dt, X_t + dX_t) - f(t, X_t) \\ &= \left[\frac{\partial f}{\partial t}(t, X_t) \;+\; \mathbf{b_t}(\mathbf{X_t})\, \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{t}, \mathbf{X_t}) \right. \\ &\qquad \left. + \frac{\mathbf{1}}{\mathbf{2}}\, \sigma_{\mathbf{t}}^2(\mathbf{X_t})\, \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}(\mathbf{t}, \mathbf{X_t}) \right] dt + \underbrace{\frac{\partial f}{\partial x}(t, X_t)\, \sigma_{\mathbf{t}}(\mathbf{X_t})\, \mathbf{dW_t}}_{\text{Martingale increment}} \end{aligned}$$

$$\Downarrow$$

Local description of the predictable increment

$$\mathbb{E}\left([f(t + dt, X_t + dX_t) - f(t, X_t)] \mid \mathcal{F}_t \right) = \left(\frac{\partial}{\partial t} + L_t \right)(f)(t, X_t)\; dt$$

with the (infinitesimal) generator

$$L_t(f) = b_t\, \frac{\partial f}{\partial x} + \frac{1}{2}\, \sigma_t^2\, \frac{\partial^2 f}{\partial x^2}$$

Second key observation ⚠

$$dM_t(f) := \frac{\partial f}{\partial x}(t, X_t)\ \sigma_t(X_t)\ dW_t$$

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$$dM_t(f) := \frac{\partial f}{\partial x}(t, X_t)\ \sigma_t(X_t)\ dW_t$$

$$\Downarrow$$

$$\mathbb{E}\left(\mathbf{dM_t(f)} \mid \mathcal{F}_\mathbf{t}\right) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{t}, \mathbf{X_t})\ \sigma_\mathbf{t}(\mathbf{X_t})\ \mathbb{E}\left([\mathbf{W_{t+dt}} - \mathbf{W_t}] \mid \mathcal{F}_\mathbf{t}\right) = \mathbf{0}$$

$$\begin{aligned}
\mathbb{E}\left((\mathbf{dM_t(f)})^\mathbf{2} \mid \mathcal{F}_\mathbf{t}\right) &= \left(\frac{\partial f}{\partial x}(t, X_t)\ \sigma_t(X_t)\right)^2\ \mathbb{E}\left([W_{t+dt} - W_t]^2 \mid \mathcal{F}_t\right) \\
&= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{t}, \mathbf{X_t})\ \sigma_\mathbf{t}(\mathbf{X_t})\right)^\mathbf{2}\ \mathbf{dt}
\end{aligned}$$

Second key observation

$$dM_t(f) := \frac{\partial f}{\partial x}(t, X_t)\ \sigma_t(X_t)\ dW_t$$

$$\Downarrow$$

$$\begin{aligned}
\mathbb{E}\left(\mathbf{dM_t(f)} \mid \mathcal{F}_t\right) &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{t}, \mathbf{X_t})\ \sigma_t(\mathbf{X_t})\ \mathbb{E}\left([\mathbf{W_{t+dt}} - \mathbf{W_t}] \mid \mathcal{F}_t\right) = \mathbf{0} \\
\mathbb{E}\left((\mathbf{dM_t(f)})^2 \mid \mathcal{F}_t\right) &= \left(\frac{\partial f}{\partial x}(t, X_t)\ \sigma_t(X_t)\right)^2\ \mathbb{E}\left([W_{t+dt} - W_t]^2 \mid \mathcal{F}_t\right) \\
&= \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{t}, \mathbf{X_t})\ \sigma_t(\mathbf{X_t})\right)^2\ \mathbf{dt}
\end{aligned}$$

$$\Downarrow$$

$M_t(f)$ **Martingale s.t.**

$$M_t^2(f) = \langle \mathbf{M(f)} \rangle_t\ +\ \textbf{martingale}$$

with the angle bracket

$$\langle \mathbf{M(f)} \rangle_t = \int_0^t \left(\frac{\partial f}{\partial x}(s, X_s)\ \sigma_s(X_s)\right)^2\ ds$$

Conclusion (cf. slide 29 - lecture slide 5)

Ito-Doeblin formula for $1d$-diffusions:

$$df(t, X_t) \quad = \quad \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\ dt \ + \ dM_t(f)$$

Conclusion (cf. slide 29 - lecture slide 5)

Ito-Doeblin formula for $1d$-diffusions:

$$df(t, X_t) = \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\, dt + dM_t(f)$$

with a martingale $M_t(f)$ with angle bracket

$$\begin{aligned} d\langle M(f)\rangle_t &= \left(\frac{\partial f}{\partial x}(t, X_t)\ \sigma_t(X_t)\right)^2 dt \\ &:= \Gamma_{L_t}(f(t,.), f(t,.))(X_t)\ dt \quad \textbf{(exercise)} \end{aligned}$$

Generators and semigroups

Conditional expectations operators

$$\forall s \leq t \qquad P_{s,t}(f)(x) = \mathbb{E}\left(f(X_t) \mid X_s = x\right) = \int \underbrace{P_{s,t}(x, dy)}_{=\mathbb{P}(X_t \in dy \ \mid \ X_s = x)} f(y)$$

Generators and semigroups

Conditional expectations operators

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For any $0 \leq r \leq s \leq t$

$$\begin{aligned} P_{r,t}(f)(X_r) &= \mathbb{E}\left(f(X_t) \mid X_r\right) \\ &= \mathbb{E}\left(\underbrace{\mathbb{E}\left(f(X_t) \mid X_s\right)}_{=P_{s,t}(f)(X_s)} \mid X_r\right) = P_{r,s}(P_{s,t}(f))(X_r) \end{aligned}$$

Generators and semigroups

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$$\Updownarrow$$

Semigroup of the stochastic process

$$\forall r \leq s \leq t \qquad P_{r,t} = P_{r,s}P_{s,t} \quad \text{and} \quad P_{t,t} = Id$$

Generators and semigroups

Conditional expectations operators

$$\forall s \leq t \qquad P_{s,t}(f)(x) = \mathbb{E}\left(f(X_t) \mid X_s = x\right) = \int \underbrace{P_{s,t}(x, dy)}_{=\mathbb{P}(X_t \in dy \mid X_s = x)} \; f(y)$$

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$$\Updownarrow$$

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$$\forall r \leq s \leq t \qquad P_{r,t} = P_{r,s}P_{s,t} \quad \text{and} \quad P_{t,t} = Id$$

$$P_{t,t+dt}(f)(x) - f(x) = \mathbb{E}\left([f(X_{t+dt}) - f(X_t)] \mid X_t = x\right) = L_t(f)(x)\; dt$$

Generators and semigroups

Conditional expectations operators

$$\forall s \leq t \qquad P_{s,t}(f)(x) = \mathbb{E}\left(f(X_t) \mid X_s = x\right) = \int \underbrace{P_{s,t}(x, dy)}_{=\mathbb{P}(X_t \in dy \mid X_s = x)} \; f(y)$$

For any $0 \leq r \leq s \leq t$

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$$\Updownarrow$$

Semigroup of the stochastic process

$$\forall r \leq s \leq t \qquad \boldsymbol{P_{r,t} = P_{r,s} P_{s,t}} \quad \text{and} \quad \boldsymbol{P_{t,t} = Id}$$

$$P_{t,t+dt}(f)(x) - f(x) = \mathbb{E}\left([f(X_{t+dt}) - f(X_t)] \mid X_t = x\right) = L_t(f)(x) \; dt$$

$$\Updownarrow$$

Taylor expansion of the semigroup

$$\boldsymbol{P_{t,t+dt} = Id + L_t \; dt + \mathrm{O}(dt)}$$

Generators and semigroups

Diffusion semigroup

$$P_{r,t} = P_{r,s}P_{s,t} \quad \text{and} \quad \boldsymbol{P_{t,t+dt} = Id + L_t\ dt + \mathrm{O}(dt)}$$

Generators and semigroups

Diffusion semigroup

$$P_{r,t} = P_{r,s}P_{s,t} \quad \text{and} \quad \boldsymbol{P_{t,t+dt} = Id + L_t\ dt + \mathrm{O}(dt)}$$

2 important consequences

$$\frac{d}{dt}P_{s,t} = \frac{1}{dt}\,[\underbrace{P_{s,t+dt}}_{=P_{s,t}P_{t,t+dt}} - P_{s,t}] = P_{s,t}\left[\frac{P_{t,t+dt} - Id}{dt}\right] = P_{s,t}L_t \qquad (3)$$

Generators and semigroups

Diffusion semigroup

$$P_{r,t} = P_{r,s}P_{s,t} \quad \text{and} \quad \boldsymbol{P_{t,t+dt} = Id + L_t\ dt + \mathrm{O}(dt)}$$

2 important consequences

$$\frac{d}{dt}P_{s,t} = \frac{1}{dt}\left[\underbrace{P_{s,t+dt}}_{=P_{s,t}P_{t,t+dt}} - P_{s,t}\right] = P_{s,t}\left[\frac{P_{t,t+dt} - Id}{dt}\right] = P_{s,t}L_t \qquad (3)$$

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Generators and semigroups

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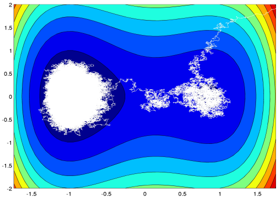
The Fokker Planck equation



Back to Ito-Doeblin formula for $1d$-diffusions

$$df(t, X_t) \;=\; \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\; dt \;+\; dM_t(f)$$

Evolution equations

$$\mathbb{E}\left(f(X_t)\right) = \int f(x)\; \mathbb{P}(\mathbf{X_t} \in \mathbf{dx}) = \int_{-\infty}^{+\infty} f(x)\; \mathbf{p_t}(\mathbf{x})\; \mathbf{dx}$$

Exercise: $\forall f$ **twice diff** $\oplus$ **all** $f^{(k)}(+/-\infty) = 0$ **for** $k = 0, 1, 2$

- **First step:**

$$d\mathbb{E}(f(X_t)) = \ldots = \mathbb{E}\left(L_t(f)(X_t)\right)\; dt$$

- **Second step:**

$$d\mathbb{E}(f(X_t)) = \ldots = \left[\int f(x)\; \frac{\partial p_t}{\partial t}(x)\; dx\right]\; dt$$

- **Third step:**

$$\mathbb{E}\left(L_t(f)(X_t)\right) = \ldots = \int f(x)\; \left(-\frac{\partial(b_t p_t)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma_t^2 p_t)}{\partial x^2}\right)\; dx$$

- **Conclusion:** ...

The equivalence principle

Diffusion processes - Ito-Doeblin formula

$$
\begin{aligned}
dX_t &= b_t(X_t)dt + \sigma_t(X_t)\ dW_t \\
&\Updownarrow \\
df(t, X_t) &= \left[\frac{\partial}{\partial t} + L_t\right](f)(t, X_t)\ dt\ +\ dM_t(f)
\end{aligned}
$$

with infinitesimal generator

$$L_t(f) = b_t\ \frac{\partial f}{\partial x} + \frac{1}{2}\ \sigma_t^2\ \frac{\partial^2 f}{\partial x^2}$$

$$\Updownarrow \quad \boldsymbol{[p_t(x)dx = \mathbb{P}(X_t \in dx)]}$$

The Fokker Planck equation (2nd order PDE)

$$\frac{\partial p_t}{\partial t} = -\frac{\partial(b_t p_t)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma_t^2 p_t)}{\partial x^2}$$

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- Monte Carlo methods based on (**the simulation of**) i.i.d. copies $(X_t^i)_{1\leq i\leq N}$ of X_t:

$$\mathbb{E}(f(X_t)) = \int\ f(x)\ \mathbb{P}(X_t \in dx) = \int_{-\infty}^{+\infty}\ f(x)\ p_t(x)\ dx \simeq \frac{\mathbf{1}}{\mathbf{N}}\sum_{\mathbf{1\leq i\leq N}}\ \mathbf{f(X_t^i)}$$

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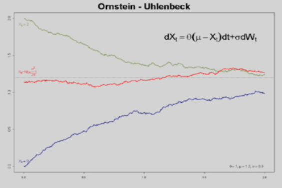
- Brownian motion, diff., martingales, sg $\rightsquigarrow$ **analysis $\oplus$ numerical solving of PDE**

Ex.1: `O.U. Process (Java applet - Ulm Univ. Physics dept.)`

Ornstein-Uhlenbeck process $(a, \sigma > 0,\ \ b \in \mathbb{R})$

$$dX_t = \ a\,(b - X_t)\ dt + \sigma\ dW_t$$



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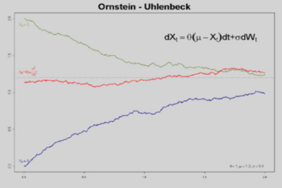

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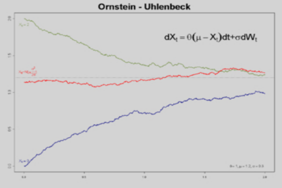

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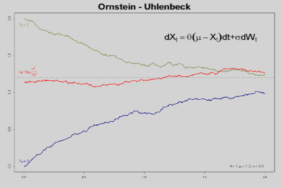

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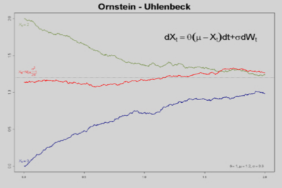

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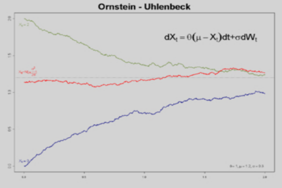

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$\mathbb{E}(X_t \mid X_0) = e^{-at}\ X_0 + b\ \left(1 - e^{-at}\right)$ $\quad$ $\mathrm{Var}(X_t \mid X_0) = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$ $\oplus$ "estimate" of (a, b, σ) ??

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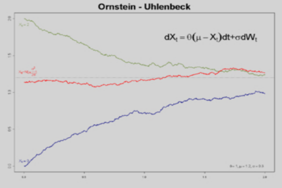

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Solution:

$$X_t - \mathbb{E}(X_t \mid X_0) = \sigma \int_0^t e^{-a(t-s)} \ dW_s \Rightarrow \mathbb{E}\left(\left[X_t - \mathbb{E}(X_t \mid X_0)\right]^2 \mid X_0\right) = \sigma^2 \int_0^t e^{-2a(t-s)} \ ds$$

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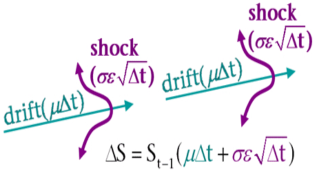

Geometric BM process $(b_t, \sigma_t \in \mathbb{R})$

$$dX_t = b_t\ X_t\ dt + \ \sigma_t\ X_t\ dW_t$$

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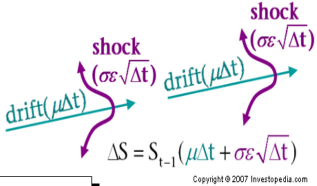

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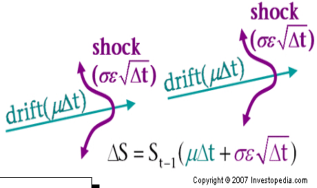

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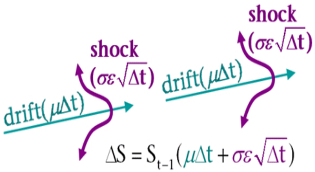

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$$\Downarrow$$

$$\log X_t - \log X_0 = \int_0^t \left(b_s - \frac{1}{2}\ \sigma_s^2 \right)\ ds + \int_0^t \sigma_s dW$$

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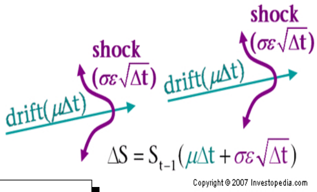

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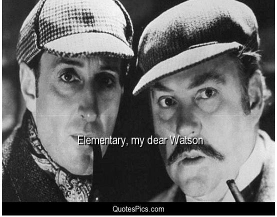
$$\Downarrow$$

$$\log X_t - \log X_0 = \int_0^t \left(b_s - \frac{1}{2}\ \sigma_s^2\right)\ ds + \int_0^t \sigma_s dW$$

$$\Downarrow$$

$$X_t = X_0\ \exp\left(\int_0^t \left(b_s - \frac{1}{2}\ \sigma_s^2\right)\ ds + \int_0^t \sigma_s\ dW_s\right)$$

An elementary case $b_t = 0$, $\sigma_t = \sigma$



$$X_t \quad = \quad X_0 \, \exp\left(\sigma \, W_t - \frac{\sigma^2}{2} \, t\right)$$

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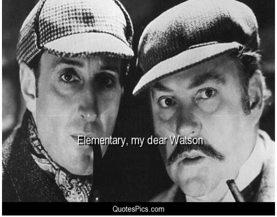
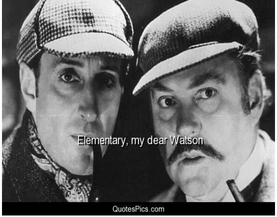


$$X_t \;=\; X_0 \;\exp\left(\sigma\, W_t - \frac{\sigma^2}{2}\, t\right) = X_s \;\exp\left(\sigma \;\; \underbrace{(W_t - W_s)}_{\overset{law}{=} W_{t-s} \sim N(0,t-s)} \;\; - \frac{\sigma^2}{2}\,(t-s)\right)$$

An elementary case $b_t = 0$, $\sigma_t = \sigma$ ⚠

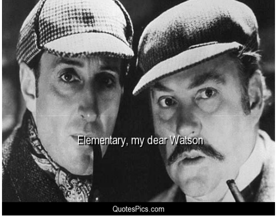


$$X_t = X_0 \exp\left(\sigma\, W_t - \frac{\sigma^2}{2}\, t\right) = X_s \exp\left(\sigma \underbrace{(W_t - W_s)}_{\stackrel{law}{=} W_{t-s} \sim N(0,t-s)} - \frac{\sigma^2}{2}\,(t-s)\right)$$

$$\Downarrow$$

$$P_{s,t}(f)(x) = \mathbb{E}\left(f\left(x\, e^V\right)\right) \quad \text{with} \quad V := \sigma\, W_{t-s} - \frac{\sigma^2}{2}\,(t-s) \sim N(\underbrace{-\frac{\sigma^2}{2}\,(t-s)}_{:=\mathbf{m}}, \underbrace{\sigma^2(t-s)}_{:=\tau^2})$$

An elementary case $b_t = 0$, $\sigma_t = \sigma$



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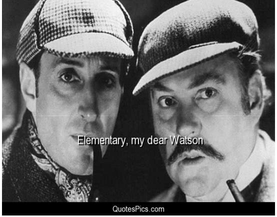
Example for $f(x) = (x-c)_+ = (x-c)\, 1_{x \geq c}$ with $v := (c/x)$:

$$\Downarrow$$

$$\begin{aligned}
x^{-1}\, P_{s,t}(f)(x) &= \mathbb{E}\left(\left(e^V - v\right)\, 1_{V \geq \log v}\right) \\
&= \mathbb{E}\left(e^V\, 1_{V \geq \log v}\right) - v\, \mathbb{E}\left(1_{V \geq \log v}\right) \\
&= \underbrace{e^{\mathbf{m} + \frac{\tau^2}{2}}}_{=1}\, \mathbb{E}\left(1_{\tau N(0,1) + \boldsymbol{m} + \tau^2 \geq \log v}\right) - v\, \mathbb{E}\left(1_{\tau N(0,1) + \boldsymbol{m} \geq \log v}\right) = \ldots
\end{aligned}$$

An elementary case $b_t = 0$, $\sigma_t = \sigma$



$$X_t \;=\; X_0 \, \exp\left(\sigma \, W_t - \frac{\sigma^2}{2} \, t\right) = X_s \, \exp\left(\sigma \, \underbrace{(W_t - W_s)}_{\overset{law}{=} W_{t-s} \sim N(0,t-s)} - \frac{\sigma^2}{2} \, (t-s)\right)$$

$$\Downarrow$$

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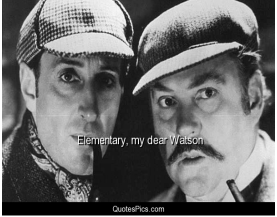
Proof:

$$\mathbb{E}\left(e^V \varphi(V)\right) \;=\; e^{\mathbf{m} + \frac{\tau^2}{2}} \; \mathbb{E}\left(\varphi(V + \tau^2)\right)$$

An elementary case $b_t = 0,\ \sigma_t = \sigma$



$$X_t \;=\; X_0\ \exp\left(\sigma\ W_t - \frac{\sigma^2}{2}\ t\right) = X_s\ \exp\left(\sigma\ \underbrace{(W_t - W_s)}_{\overset{law}{=} W_{t-s} \sim N(0,t-s)}\ - \frac{\sigma^2}{2}\ (t-s)\right)$$

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$$P_{s,t}(f)(x) = \mathbb{E}\left(f\left(x\ e^V\right)\right) \quad \text{with} \quad V := \sigma\ W_{t-s} - \frac{\sigma^2}{2}\ (t-s) \sim\ N(\underbrace{-\frac{\sigma^2}{2}\ (t-s)}_{:=\mathbf{m}}, \underbrace{\sigma^2(t-s)}_{:=\tau^2})$$

Example for $f(x) = (x-c)_+ = (x-c)\ 1_{x \geq c}$ **with** $v := (c/x)$**:**

$$\Downarrow$$

$$\boxed{\begin{aligned} x^{-1}\ P_{s,t}(f)(x) &= \mathbb{E}\left(\left(e^V - v\right)\ 1_{V \geq \log v}\right) \\ &= \mathbb{E}\left(e^V\ 1_{V \geq \log v}\right) - v\ \mathbb{E}\left(1_{V \geq \log v}\right) \\ &= \underbrace{e^{\mathbf{m}+\frac{\tau^2}{2}}}_{=1}\ \mathbb{E}\left(1_{\boldsymbol{\tau} N(0,1)+\boldsymbol{m}+\boldsymbol{\tau}^2 \geq \log v}\right) - v\ \mathbb{E}\left(1_{\boldsymbol{\tau} N(0,1)+\boldsymbol{m} \geq \log v}\right) = \ldots \end{aligned}}$$

Proof:

$$\mathbb{E}\left(e^V \varphi(V)\right) \;=\; e^{\mathbf{m}+\frac{\tau^2}{2}}\ \mathbb{E}\left(\varphi(V+\tau^2)\right) = e^{\boldsymbol{m}+\frac{\tau^2}{2}}\,\mathbb{E}\left(\varphi(\boldsymbol{\tau} N(0,1) + \boldsymbol{m} + \boldsymbol{\tau}^2)\right)$$

using

$$-\frac{1}{2\tau^2}\ (v-m)^2 + v = -\frac{1}{2\tau^2}\ \left(v - \left(m+\tau^2\right)\right)^2\ +\ \left(m + \frac{\tau^2}{2}\right)$$

Application - Math - Finance 1/4

$$\left\{ \begin{array}{lcl} d\mathcal{S}_t^{(0)} & = & r_t\ \mathcal{S}_t^{(0)}\ dt \quad \text{"reference" cash-flow with riskless return rate } r_t \\ d\mathcal{S}_t & = & b_t\ \mathcal{S}_t\ dt + \sigma_t\ \mathcal{S}_t\ dW_t \quad \text{risky asset with return rate } b_t, \text{ volatility } \sigma_t. \end{array} \right.$$

$$\Downarrow$$

The deflated risky asset

$$\overline{\mathcal{S}}_t \quad = \quad \mathcal{S}_t/\mathcal{S}_t^{(0)} = e^{-\int_0^t \ r_s ds}\ \mathcal{S}_t/\mathcal{S}_t^{(0)} = \overline{\mathcal{S}}_0\ \exp\left(\int_0^t \left([b_s - r_s] - \frac{1}{2}\ \sigma_s^2\right)\ ds + \int_0^t \sigma_s\ dW_s\right)$$

Application - Math - Finance 1/4

$$\begin{cases} d\mathcal{S}_t^{(0)} &= r_t\ \mathcal{S}_t^{(0)}\ dt \quad \text{"reference" cash-flow with riskless return rate } r_t \\ d\mathcal{S}_t &= b_t\ \mathcal{S}_t\ dt + \sigma_t\ \mathcal{S}_t\ dW_t \quad \text{risky asset with return rate } b_t, \text{ volatility } \sigma_t. \end{cases}$$

$$\Downarrow$$

The deflated risky asset

$$\overline{\mathcal{S}}_t = \mathcal{S}_t/\mathcal{S}_t^{(0)} = e^{-\int_0^t r_s ds}\ \mathcal{S}_t/\mathcal{S}_t^{(0)} = \overline{\mathcal{S}}_0\ \exp\left(\int_0^t \left([b_s - r_s] - \frac{1}{2}\ \sigma_s^2\right)\ ds + \int_0^t \sigma_s\ dW_s\right)$$

Important observation:

$$\overline{\mathcal{S}}_t \quad \textbf{Martingale} \quad \Longleftrightarrow \quad b_t = r_t \quad \Longleftrightarrow \quad d\overline{\mathcal{S}}_t = \sigma_t\ \overline{\mathcal{S}}_t dW_t$$

Application - Math - Finance 2/4

Case $b_t = r_t = r$ and $\mathcal{S}_t^{(0)} = e^{rt} \times 1$ ($\Longrightarrow$ $\overline{\mathcal{S}}_t$ Martingale !):

Application - Math - Finance 2/4

Case $b_t = r_t = r$ and $\mathcal{S}_t^{(0)} = e^{rt} \times 1$ ($\Longrightarrow$ $\overline{\mathcal{S}}_t$ Martingale !):

Self-financing portfolio $\left(p_t^{(0)}, p_t\right)$

$$\mathcal{P}_t = \underbrace{p_{t-dt}^{(0)}\ \mathcal{S}_t^{(0)} + p_{t-dt}\ \mathcal{S}_t}_{\textit{value of the portfolio at time } t}$$

Application - Math - Finance 2/4

Case $b_t = r_t = r$ **and** $\mathcal{S}_t^{(0)} = e^{rt} \times 1$ **(**$\Longrightarrow \quad \overline{\mathcal{S}}_t$ **Martingale !):**

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Application - Math - Finance 2/4

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$$\Downarrow$$

$$d\mathcal{P}_t = \mathcal{P}_{t+dt} - \mathcal{P}_t$$

Application - Math - Finance 2/4

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$$\Downarrow$$

$$d\mathcal{P}_t = \mathcal{P}_{t+dt} - \mathcal{P}_t = \left[p_t^{(0)}\ \mathcal{S}_{t+dt}^{(0)} + p_t\ \mathcal{S}_{t+dt}\right]$$

Application - Math - Finance 2/4

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Application - Math - Finance 2/4

Case $b_t = r_t = r$ **and** $\mathcal{S}_t^{(0)} = e^{rt} \times 1$ **(**$\Longrightarrow \quad \overline{\mathcal{S}}_t$ **Martingale !):**

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$$\Downarrow$$

$$\begin{aligned} d\mathcal{P}_t &= \mathcal{P}_{t+dt} - \mathcal{P}_t = \left[p_t^{(0)}\ \mathcal{S}_{t+dt}^{(0)} + p_t\ \mathcal{S}_{t+dt}\right] - \left[p_t^{(0)}\ \mathcal{S}_t^{(0)} + p_t\ \mathcal{S}_t\right] \\ &= p_t^{(0)}\ \left(\mathcal{S}_{t+dt}^{(0)} - \mathcal{S}_t^{(0)}\right) + p_t\ \left(\mathcal{S}_{t+dt} - \mathcal{S}_t\right) = p_t^{(0)}\ d\mathcal{S}_t^{(0)} + p_t\ d\mathcal{S}_t \end{aligned}$$

Application - Math - Finance 2/4

Case $b_t = r_t = r$ and $\mathcal{S}_t^{(0)} = e^{rt} \times 1$ ($\Longrightarrow$ $\overline{\mathcal{S}}_t$ Martingale !):

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Deflated Self-financing portfolio

$$\begin{aligned} \overline{\mathcal{P}}_{t+dt} &:= e^{-r(t+dt)}\ \mathcal{P}_{t+dt} &&= p_t^{(0)} + p_t\ \overline{\mathcal{S}}_{t+dt} \\ \overline{\mathcal{P}}_t &:= e^{-rt}\ \mathcal{P}_t &&= p_t^{(0)} + p_t\ \overline{\mathcal{S}}_t \end{aligned}$$

Application - Math - Finance 2/4

Case $b_t = r_t = r$ **and** $\mathcal{S}_t^{(0)} = e^{rt} \times 1$ **(**$\Longrightarrow \quad \overline{\mathcal{S}}_t$ **Martingale !):**

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$$\mathcal{P}_t = \underbrace{p_{t-dt}^{(0)}\ \mathcal{S}_t^{(0)} + p_{t-dt}\ \mathcal{S}_t}_{\text{value of the portfolio at time } t} = \underbrace{p_t^{(0)}\ \mathcal{S}_t^{(0)} + p_t\ \mathcal{S}_t}_{\text{choice of a (self financed) new strategy}}$$

$$\Downarrow$$

$$\begin{aligned} d\mathcal{P}_t &= \mathcal{P}_{t+dt} - \mathcal{P}_t = \left[p_t^{(0)}\ \mathcal{S}_{t+dt}^{(0)} + p_t\ \mathcal{S}_{t+dt}\right] - \left[p_t^{(0)}\ \mathcal{S}_t^{(0)} + p_t\ \mathcal{S}_t\right] \\ &= p_t^{(0)}\ \left(\mathcal{S}_{t+dt}^{(0)} - \mathcal{S}_t^{(0)}\right) + p_t\ \left(\mathcal{S}_{t+dt} - \mathcal{S}_t\right) = p_t^{(0)}\ d\mathcal{S}_t^{(0)} + p_t\ d\mathcal{S}_t \end{aligned}$$

Deflated Self-financing portfolio

$$\begin{array}{rcccl} \overline{\mathcal{P}}_{t+dt} & := & e^{-r(t+dt)}\ \mathcal{P}_{t+dt} & = & p_t^{(0)} + p_t\ \overline{\mathcal{S}}_{t+dt} \\ \overline{\mathcal{P}}_t & := & e^{-rt}\ \mathcal{P}_t & = & p_t^{(0)} + p_t\ \overline{\mathcal{S}}_t \end{array}$$

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$$d\overline{\mathcal{P}}_t = \overline{\mathcal{P}}_{t+dt} - \overline{\mathcal{P}}_t$$

Application - Math - Finance 2/4

Case $b_t = r_t = r$ and $\mathcal{S}_t^{(0)} = e^{rt} \times 1$ ($\Longrightarrow$ $\overline{\mathcal{S}}_t$ Martingale !):

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$$\Downarrow$$

$$\begin{aligned} d\mathcal{P}_t &= \mathcal{P}_{t+dt} - \mathcal{P}_t = \left[p_t^{(0)}\ \mathcal{S}_{t+dt}^{(0)} + p_t\ \mathcal{S}_{t+dt}\right] - \left[p_t^{(0)}\ \mathcal{S}_t^{(0)} + p_t\ \mathcal{S}_t\right] \\ &= p_t^{(0)}\ \left(\mathcal{S}_{t+dt}^{(0)} - \mathcal{S}_t^{(0)}\right) + p_t\ \left(\mathcal{S}_{t+dt} - \mathcal{S}_t\right) = p_t^{(0)}\ d\mathcal{S}_t^{(0)} + p_t\ d\mathcal{S}_t \end{aligned}$$

Deflated Self-financing portfolio

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$$\Downarrow$$

$d\overline{\mathcal{P}}_t = \overline{\mathcal{P}}_{t+dt} - \overline{\mathcal{P}}_t = p_t\ \mathbf{d\overline{\mathcal{S}}_t}$ = **Martingale increment**

Martingale design 3/4

$$d\overline{\mathcal{S}}_t = \sigma_t \, \overline{\mathcal{S}}_t dW_t$$

$\Downarrow$ **cf. slide 14**

Martingale design

$$s \mapsto P_{s,t}(f_t)(\overline{\mathcal{S}}_s) \; = \; \textbf{Martingale on } [0,t] \textbf{ with } \underline{\textbf{terminal condition}} \quad \boldsymbol{f_t(\overline{\mathcal{S}}_t)}$$

Evolution equation:

$$dP_{s,t}(f_t)(\overline{\mathcal{S}}_s) = \frac{\partial P_{s,t}(f_t)}{\partial x}(\overline{\mathcal{S}}_s)\, \sigma_s \, \overline{\mathcal{S}}_s \;\; dW_s = \boldsymbol{\frac{\partial P_{s,t}(f_t)}{\partial x}(\overline{\mathcal{S}}_s)\; d\overline{\mathcal{S}}_s}$$

Martingale design 3/4

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Deflated Self-financing portfolio is also a martingale

$$d\overline{\mathcal{P}}_s = \boldsymbol{p_s}\;\mathbf{d}\overline{\boldsymbol{\mathcal{S}}}_s = \text{Martingale increment}$$

Martingale design 3/4

$$d\overline{\mathcal{S}}_t = \sigma_t\, \overline{\mathcal{S}}_t dW_t$$

$\Downarrow$ **cf. slide 14**

Martingale design

$s \mapsto P_{s,t}(f_t)(\overline{\mathcal{S}}_s) \;=\;$ **Martingale on $[0, t]$ with terminal condition** $\;\boldsymbol{f_t(\overline{\mathcal{S}}_t)}$

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Deflated Self-financing portfolio is also a martingale

$$d\overline{\mathcal{P}}_s = \boldsymbol{p_s}\; \mathbf{d}\overline{\boldsymbol{\mathcal{S}}}_s = \text{Martingale increment}$$

$$\Downarrow$$

$$\forall \boldsymbol{s} \in [\mathbf{0}, \boldsymbol{t}] \qquad \boldsymbol{p_s} = \frac{\partial \boldsymbol{P_{s,t}(f_t)}}{\partial \boldsymbol{x}}(\overline{\boldsymbol{\mathcal{S}}}_s) \implies \textbf{drives the portfolio } \overline{\boldsymbol{\mathcal{P}}}_s := \boldsymbol{P_{s,t}(f_t)(\overline{\mathcal{S}}_s)} \textbf{ to}$$

Martingale design 3/4

$$d\overline{\mathcal{S}}_t = \sigma_t\, \overline{\mathcal{S}}_t dW_t$$

$\Downarrow$ **cf. slide 14**

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$s \mapsto P_{s,t}(f_t)(\overline{\mathcal{S}}_s) \;=\;$ **Martingale on $[0,t]$ with terminal condition** $\;\boldsymbol{f_t(\overline{\mathcal{S}}_t)}$

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Deflated Self-financing portfolio is also a martingale

$$d\overline{\mathcal{P}}_s = p_s\; \mathbf{d}\overline{\mathcal{S}}_s = \text{Martingale increment}$$

$$\Downarrow$$

$$\forall s \in [0,t] \qquad p_s = \frac{\partial P_{s,t}(f_t)}{\partial x}(\overline{\mathcal{S}}_s) \Longrightarrow \text{ drives the portfolio } \overline{\mathcal{P}}_s := P_{s,t}(f_t)(\overline{\mathcal{S}}_s) \text{ to } \;\overline{\mathcal{P}}_t = f_t(\overline{\mathcal{S}}_t)$$

The Black -Scholes -Merton formula 4/4

Security contracts $=$ Right $\underbrace{\text{to buy}}_{\text{Call option}}$ or $\underbrace{\text{sell}}_{\text{Put option}}$ shares at a given price

The Black -Scholes -Merton formula 4/4

Security contracts = Right $\underbrace{\textbf{to buy}}_{\text{Call option}}$ **or** $\underbrace{\textbf{sell}}_{\text{Put option}}$ **shares at a given price**

Example: European style options

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The Black -Scholes -Merton formula 4/4

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Replicating portfolio - Risk elimination :

$$\overline{\mathcal{P}}_s := P_{s,t}(f_t)(\overline{\mathcal{S}}_s) = \underbrace{P_{0,t}(f_t)(\overline{\mathcal{S}}_0)}_{:=\overline{\mathcal{P}}_0} + \int_0^s \underbrace{\frac{\partial P_{r,t}(f_t)}{\partial x}(\overline{\mathcal{S}}_r)}_{:=p_r}\ d\overline{\mathcal{S}}_r$$

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Elementary case - Geometric BM:

$\sigma_t = \sigma \Rightarrow \overline{\mathcal{S}}_t = \overline{\mathcal{S}}_0\ \exp\left(\sigma\ W_t - \frac{\sigma^2}{2}\ t\right) \Rightarrow$ **Explicit Gaussian formulae slide 19 = "Black-Scholes formula"**

`Youtube documentary (Nobel price 1997)`