Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes 2

Consultations (RC 5112):

Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

Citation of the day

As far as the laws of mathematics refer to reality, they are not certain,

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personal question

$$X \text{ random variable} \Leftrightarrow \quad \mathrm{Law}(X) = \textbf{certain} \text{ ??}$$

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personal question

$$X \text{ random variable} \Leftrightarrow \quad \mathrm{Law}(X) = \textbf{certain} \text{ ??}$$

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

– Hilbert, David (1862-1943)

Some basic notation :-) = 

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$$\forall 1 \leq i \leq d \qquad \underbrace{\mathbb{P}(Y=j)}_{=p_Y(j)} = \sum_{1\leq i\leq d} \underbrace{\mathbb{P}(X=i)}_{=p_X(i)} \; \underbrace{\mathbb{P}(Y=j \mid X=i)}_{=M(i,j)}$$

Some basic notation :-) = $\ddot{\smile}$

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$$\Updownarrow$$

Matrix notation:

$$p_Y = [\mathbb{P}(Y=1), \ldots, \mathbb{P}(Y=d)]$$

$$= \underbrace{[\mathbb{P}(X=1), \ldots, \mathbb{P}(X=d)]}_{=p_X} \times \underbrace{\begin{pmatrix} \mathbb{P}(Y=1 \mid X=1) & \mathbb{P}(Y=2 \mid X=1)) & \ldots & \mathbb{P}(Y=d \mid X=1) \\ \mathbb{P}(Y=1 \mid X=2) & \mathbb{P}(Y=2 \mid X=2)) & \ldots & \mathbb{P}(Y=d \mid X=2) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{P}(Y=1 \mid X=d) & \mathbb{P}(Y=2 \mid X=d)) & \ldots & \mathbb{P}(Y=d \mid X=d) \end{pmatrix}}_{M=(M(i,j))_{i,j}}$$

Some basic notation :-) = ⌣

$$\forall 1 \le i \le d \qquad \underbrace{\mathbb{P}(Y = j)}_{=p_Y(j)} = \sum_{1 \le i \le d} \underbrace{\mathbb{P}(X = i)}_{=p_X(i)} \; \underbrace{\mathbb{P}(Y = j \mid X = i)}_{=M(i,j)}$$

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$$\Updownarrow$$

Matrix synthetic notation:

$$p_Y = p_X M$$

Some basic notation

$$\mathbb{E}(f(Y) \mid X = i) = \sum_{1 \le j \le d} \underbrace{\mathbb{P}(Y = j \mid X = i)}_{=M(i,j)} \, f(j)$$

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Matrix notation:

$$\underbrace{\begin{pmatrix} \mathbb{P}(Y=1 \mid X=1) & \mathbb{P}(Y=2 \mid X=1)) & \dots & \mathbb{P}(Y=d \mid X=1) \\ \mathbb{P}(Y=1 \mid X=2) & \mathbb{P}(Y=2 \mid X=2)) & \dots & \mathbb{P}(Y=d \mid X=2) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbb{P}(Y=1 \mid X=d) & \mathbb{P}(Y=2 \mid X=d)) & \dots & \mathbb{P}(Y=d \mid X=d) \end{pmatrix}}_{M=(M(i,j))_{i,j}} \underbrace{\begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(d) \end{pmatrix}}_{=f} = \underbrace{\begin{pmatrix} \mathbb{E}(f(Y) \mid X=1) \\ \mathbb{E}(f(Y) \mid X=2) \\ \vdots \\ \mathbb{E}(f(Y) \mid X=d) \end{pmatrix}}_{=M(f)}$$

Some basic notation

$$\mathbb{E}(f(Y) \mid X = i) = \sum_{1 \le j \le d} \underbrace{\mathbb{P}(Y = j \mid X = i)}_{=M(i,j)} \; f(j)$$

$$\Updownarrow$$

Matrix notation:

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$$\Updownarrow$$

Matrix synthetic notation:

$$\mathbb{E}\left(f(Y) \mid X = i\right) = M(f)(i)$$

Some basic notation

Markov chain = "sequence of r.v."

$$X_0 \rightsquigarrow X_1 \rightsquigarrow \ldots \rightsquigarrow X_{n-1} \rightsquigarrow X_n$$

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$$\Updownarrow$$

$$\underbrace{\mathbb{P}(X_n = j)}_{=p_{X_n}(j)} = \sum_{1 \leq i \leq d} \underbrace{\mathbb{P}(X_{n-1} = i)}_{=p_{X_{n-1}}(i)} \; \underbrace{\mathbb{P}(X_n = j \mid X_{n-1} = i)}_{=M_n(i,j)}$$

Some basic notation

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$$\Updownarrow$$

Matrix synthetic notation:

$$p_{X_n} = p_{X_{n-1}} M_n = \ldots = p_{X_0} M_1 M_2 \ldots M_n$$

Some basic notation

$$\begin{aligned}\mathbb{E}\left(f(X_n) \mid X_0 = i\right) &= \mathbb{E}\left(\overbrace{\mathbb{E}\left(f(X_n) \mid X_{n-1}\right)}^{M_n(f)(X_{n-1})} \mid X_0 = i\right) \\ &= \mathbb{E}\left(M_n(f)(X_{n-1}) \mid X_0 = i\right)\end{aligned}$$

Some basic notation

$$
\begin{aligned}
\mathbb{E}\left(f(X_n) \mid X_0 = i\right) &= \mathbb{E}\left(\overbrace{\mathbb{E}\left(f(X_n) \mid X_{n-1}\right)}^{M_n(f)(X_{n-1})} \mid X_0 = i\right) \\
&= \mathbb{E}\left(M_n(f)(X_{n-1}) \mid X_0 = i\right) \\
&= \mathbb{E}\left(M_{n-1}\left(M_n(f)\right)\left(X_{n-2}\right) \mid X_0 = i\right)
\end{aligned}
$$

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&= \mathbb{E}\left(M_{n-1}\left(M_n(f)\right)(X_{n-2}) \mid X_0 = i\right) \\
&= \ldots \\
&= \mathbb{E}\left(\left(M_1 \ldots \left(M_n(f)\right)\right)(X_0) \mid X_0 = i\right)
\end{aligned}
$$

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$$
\begin{aligned}
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&= \ldots \\
&= \mathbb{E}\left(\left(M_1 \ldots \left(M_n(f)\right)\right)\left(X_0\right) \mid X_0 = i\right) \\
&= (M_1 M_2 \ldots M_n)(f)(i)
\end{aligned}
$$

Stabilizing populations - Migration processes

- 193 countries (UN report 2013) c_i, $i = 1, \ldots, 193$.
- $q_n(i)$ = average-population of country c_i at some time n (years/months/...).
- $M_n(i,j)$ = proportions of migrants from c_i to c_j at time n.

Some questions:

- Stabilization $\exists$? $q_\infty(i)$ invariant w.r.t. migration process
- Chance for two migrants to meet in some country?

Migration - Stochastic process

$$\{\overbrace{I^1_{\mathbf{i},\mathbf{n}}, I^2_{\mathbf{i},\mathbf{n}}, I^3_{\mathbf{i},\mathbf{n}}, \ldots, I^{m_{\mathbf{n}}(i)}_{\mathbf{i},\mathbf{n}}}^{\text{individuals}}\} = \text{Country } c_{\mathbf{i}} \text{ at time } n \text{ with pop. } m_{\mathbf{n}}(i)$$

During the migration process

Each $I^k_{\mathbf{i},\mathbf{n}}$ chooses the index $\widehat{I^k_{\mathbf{i},\mathbf{n}}} = \mathbf{j}$ of a country $c_{\mathbf{j}} \sim M_{\mathbf{n}}(\mathbf{i}, \mathbf{j})$

Simulation?

Migration - Stochastic process

$$\overbrace{\{I^1_{\mathbf{i},\mathbf{n}}, I^2_{\mathbf{i},\mathbf{n}}, I^3_{\mathbf{i},\mathbf{n}}, \ldots, I^{m_{\mathbf{n}}(i)}_{\mathbf{i},\mathbf{n}}\}}^{\text{individuals}} = \text{Country } c_{\mathbf{i}} \text{ at time } n \text{ with pop. } m_{\mathbf{n}}(i)$$

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Simulation? Blackboard

Migration - Stochastic process

$$\{\overbrace{I^1_{i,n}, I^2_{i,n}, I^3_{i,n}, \ldots, I^{m_n(i)}_{i,n}}^{\text{individuals}}\} = \text{Country } c_i \text{ at time } n \text{ with pop. } m_n(i)$$

During the migration process

Each $I^k_{i,n}$ chooses the index $\widehat{I}^k_{i,n} = j$ of a country $c_j \sim M_n(i, j)$

Simulation?

Blackboard

$$m_{(n+1)}(i,j) = \sum_{1 \leq k \leq m_n(i)} 1_j\left(\widehat{I}^k_{i,n}\right) = \text{Migrants } i \rightsquigarrow j$$

Migration - Stochastic process

$$\{\underbrace{I^1_{i,n}, I^2_{i,n}, I^3_{i,n}, \ldots, I^{m_n(i)}_{i,n}}_{\text{individuals}}\} = \text{Country } c_i \text{ at time } n \text{ with pop. } m_n(i)$$

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$$\Downarrow$$

$$m_{(n+1)}(j) = \sum_{1 \leq i \leq 193} m_{(n+1)}(i, j)$$

Migration - Stochastic process

$$\{\underbrace{I^1_{\mathbf{i},\mathbf{n}}, I^2_{\mathbf{i},\mathbf{n}}, I^3_{\mathbf{i},\mathbf{n}}, \ldots, I^{m_{\mathbf{n}}(i)}_{\mathbf{i},\mathbf{n}}}_{\text{individuals}}\} = \text{Country } c_{\mathbf{i}} \text{ at time } n \text{ with pop. } m_{\mathbf{n}}(i)$$

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Each $I^k_{\mathbf{i},\mathbf{n}}$ chooses the index $\widehat{I}^k_{\mathbf{i},\mathbf{n}} = \mathbf{j}$ of a country $c_{\mathbf{j}} \sim M_{\mathbf{n}}(\mathbf{i}, \mathbf{j})$

Simulation?

$$m_{(\mathbf{n}+1)}(\mathbf{i}, \mathbf{j}) = \sum_{1 \leq k \leq m_{\mathbf{n}}(i)} 1_{\mathbf{j}}\left(\widehat{I}^k_{\mathbf{i},\mathbf{n}}\right) = \text{Migrants } \mathbf{i} \rightsquigarrow \mathbf{j}$$

$$\Downarrow$$

$$m_{(\mathbf{n}+1)}(\mathbf{j}) = \sum_{1 \leq \mathbf{i} \leq 193} m_{(\mathbf{n}+1)}(\mathbf{i}, \mathbf{j})$$

If no birth & death!

Migration - Stochastic process

$$\overbrace{\{I_{i,n}^{1}, I_{i,n}^{2}, I_{i,n}^{3}, \ldots, I_{i,n}^{m_n(i)}\}}^{\text{individuals}} = \text{Country } c_i \text{ at time } n \text{ with pop. } m_n(i)$$

During the migration process

Each $I_{i,n}^{k}$ chooses the index $\widehat{I}_{i,n}^{k} = j$ of a country $c_j \sim M_n(i,j)$

Simulation?

Blackboard

$$m_{(n+1)}(i,j) = \sum_{1 \leq k \leq m_n(i)} 1_j\left(\widehat{I}_{i,n}^{k}\right) = \text{Migrants } i \rightsquigarrow j$$

$$\Downarrow$$

$$m_{(n+1)}(j) = \sum_{1 \leq i \leq 193} m_{(n+1)}(i,j)$$ **If no birth & death!**

Mean-average?

Migration - Stochastic process

$$\overbrace{\{I^1_{\mathbf{i},\mathbf{n}}, I^2_{\mathbf{i},\mathbf{n}}, I^3_{\mathbf{i},\mathbf{n}}, \ldots, I^{m_{\mathbf{n}}(i)}_{\mathbf{i},\mathbf{n}}\}}^{\text{individuals}} = \text{Country } c_{\mathbf{i}} \text{ at time } n \text{ with pop. } m_{\mathbf{n}}(i)$$

During the migration process

Each $I^k_{\mathbf{i},\mathbf{n}}$ chooses the index $\widehat{I}^k_{\mathbf{i},\mathbf{n}} = \mathbf{j}$ of a country $c_{\mathbf{j}} \sim M_{\mathbf{n}}(\mathbf{i}, \mathbf{j})$

Simulation? Blackboard

$$m_{(\mathbf{n}+\mathbf{1})}(\mathbf{i}, \mathbf{j}) = \sum_{1 \le k \le m_{\mathbf{n}}(i)} 1_{\mathbf{j}}\left(\widehat{I}^k_{\mathbf{i},\mathbf{n}}\right) = \text{Migrants } \mathbf{i} \rightsquigarrow \mathbf{j}$$

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$$m_{(\mathbf{n}+\mathbf{1})}(\mathbf{j}) = \sum_{1 \le \mathbf{i} \le 193} m_{(\mathbf{n}+\mathbf{1})}(\mathbf{i}, \mathbf{j})$$ **If no birth & death!**

Mean-average? Blackboard

Migration - Stochastic process

$$\mathbb{E}(m_n(j)) = q_n(j) \quad \Longrightarrow \quad q_n(j) = \sum_{1 \leq i \leq 193} q_{n-1}(i)\ M_n(i,j)$$

Migration - Stochastic process

$$\mathbb{E}(m_n(j)) = q_n(j) \implies q_n(j) = \sum_{1 \leq i \leq 193} q_{n-1}(i)\ M_n(i,j)$$

$$\iff q_n = q_{n-1} M_n$$

Migration - Stochastic process

$$\mathbb{E}(m_n(j)) = q_n(j) \quad \Longrightarrow \quad q_n(j) = \sum_{1 \le i \le 193} q_{n-1}(i)\ M_n(i,j)$$

$$\Longleftrightarrow \quad q_n = q_{n-1} M_n = q_0 M_1 M_2 \dots M_n$$

If World pop. size $N_n = N$ fixed:

$$\frac{q_n(i)}{N} := p_n(i) = \text{Proba on } \{1, \dots, 193\}$$

Migration - Stochastic process

$$\mathbb{E}(m_n(j)) = q_n(j) \quad \Longrightarrow \quad q_n(j) = \sum_{1\leq i\leq 193} q_{n-1}(i)\ M_n(i,j)$$
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If World pop. size $N_n = N$ fixed:

$$\frac{q_n(i)}{N} := p_n(i) = \text{Proba on } \{1,\ldots,193\} := \mathbb{P}\left(X_n = i\right)$$

Stochastic model for a migrant X_n between countries

$$\underbrace{\mathbb{P}\left(X_n = \mathbf{j}\right)}_{=p_n(\mathbf{j})} = \sum_{1\leq \mathbf{i}\leq 193} \underbrace{\mathbb{P}\left(X_{n-1} = \mathbf{i}\right)}_{=p_{n-1}(\mathbf{i})} \underbrace{\mathbb{P}\left(X_n = \mathbf{j} \mid X_{n-1} = \mathbf{i}\right)}_{=M_n(\mathbf{i},\mathbf{j})}$$

Migration - Stochastic process

$$\mathbb{E}(m_n(j)) = q_n(j) \quad \Longrightarrow \quad q_n(j) = \sum_{1\leq i\leq 193} q_{n-1}(i)\ M_n(i,j)$$
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$$\Updownarrow$$

$$p_n = p_{n-1} M_n$$

Migration - Stabilization $M_n = M$

$$p_n = p_{n-1} M \longrightarrow_{n \uparrow \infty} \; p_\infty = p_\infty M = \textbf{left eigenvector of } M$$

$$\Updownarrow$$

Stationary population $\quad q_\infty = N \times p_\infty$

Migration - Stabilization $M_n = M$

$$p_n = p_{n-1}M \longrightarrow_{n\uparrow\infty} \; p_\infty = p_\infty M = \textbf{left eigenvector of } M$$

$$\Updownarrow$$

$$\text{Stationary population} \quad q_\infty = N \times p_\infty \quad \underline{\textbf{If no birth \& death!}}$$

Migration - Stabilization $M_n = M$

$$p_n = p_{n-1} M \longrightarrow_{n\uparrow\infty} p_\infty = p_\infty M = \textbf{left eigenvector of } M$$

$$\Updownarrow$$

Stationary population $\quad q_\infty = N \times p_\infty \quad$ **If no birth & death!**

- **Power method** $M^n(\mathbf{i},\mathbf{j}) \rightarrow_{n\uparrow\infty} p_\infty(\mathbf{j})$

$$p_n = p_{n-1} M = p_{n-2} M^2 = \ldots = p_0 M^n$$

$$\Downarrow$$

$$p_n(\mathbf{j}) = \sum_{\mathbf{i}} p_0(\mathbf{i}) \underbrace{M^n(\mathbf{i},\mathbf{j})}_{\rightarrow_{n\uparrow\infty} p_\infty(\mathbf{j})} \rightarrow_{n\uparrow\infty} p_\infty(\mathbf{j})$$

- **Law of large numbers = Ergodic theorem (admitted today) = by simulation**

$$\textbf{proportions of visits to } c_{\mathbf{j}} = \frac{1}{n} \sum_{1\leq k\leq n} 1_{c_{\mathbf{j}}}(X_k) \rightarrow_{n\uparrow\infty} p_\infty(\mathbf{j})$$

The evolution of 2 migrants

Walker X_n *starting at* $X_0 = \mathbf{i}$ *& Walker* X'_n *starting at* $X'_0 = \mathbf{i}'$

$$\begin{array}{rcl} p_n(j) & = & \mathbb{P}(X_n = j) = p_0 M^n(j) \quad \text{with} \quad p_0(j) = 1_{\mathbf{i}}(j) \\ p'_n(j) & = & \mathbb{P}(X'_n = j) = p'_0 M^n(j) \quad \text{with} \quad p'_0(j) = 1_{\mathbf{i}'}(j) \end{array}$$

Natural questions:

- Do they forget their initial state?
- Can we define/couple their random evolution in the same probability space?
- What is their meeting time probabilities?

Forgetting their original country

$$p_n = p_{n-1} M \quad \oplus \quad \text{Hypothesis} \quad M(i,j) \geq \epsilon \underbrace{\lambda(i)}_{=1/193}$$

KEY ϵ-transition

$$M_\epsilon(i,j) = \frac{M(i,j) - \epsilon\lambda(j)}{1-\epsilon}$$

Forgetting their original country

$$p_n = p_{n-1} M \quad \oplus \quad \text{Hypothesis} \quad M(i,j) \geq \epsilon \underbrace{\lambda(i)}_{=1/193}$$

KEY ϵ-transition

$$M_\epsilon(i,j) = \frac{M(i,j) - \epsilon\lambda(j)}{1-\epsilon} \quad \Leftrightarrow \quad M(i,j) = (1-\epsilon)\ M_\epsilon(i,j) + \epsilon\ \lambda(j)$$

$$\Longrightarrow \quad pM = (1-\epsilon)\ pM_\epsilon + \epsilon\lambda$$

$$\Longrightarrow \quad [p - p']M = (1-\epsilon)\ [p-p']M_\epsilon$$

$$\Downarrow$$

$$\begin{aligned} \mathbf{p_{n+1}} - \mathbf{p'_{n+1}} = [p_n - p'_n]M &= (1-\epsilon)\ [\mathbf{p_n} - \mathbf{p'_n}]M_\epsilon \\ &= (1-\epsilon)^2\ [\mathbf{p_{n-1}} - \mathbf{p'_{n-1}}]M_\epsilon^2 \\ &= (1-\epsilon)^{n+1}\ [\mathbf{p_0} - \mathbf{p'_0}]M_\epsilon^{n+1} \downarrow_{n\uparrow\infty} 0 \end{aligned}$$

Coupling the 2 migrations

- Coupling 2 r.v.

$\Leftrightarrow$ Defined using the "same" randomness.

- How to couple two individuals?

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$\Leftrightarrow$ Defined using the "same" randomness.

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Why? $\rightsquigarrow$ An illustration:

$$
\begin{aligned}
\mathbb{P}(X \in A) - \mathbb{P}(Y \in A) &= \mathbb{P}(X = Y \in A,\ X = Y) + \mathbb{P}(X \in A,\ X \neq Y) \\
&\quad - \mathbb{P}(Y = X \in A,\ Y = X) - \mathbb{P}(Y \in A,\ X \neq Y) \\
&= \mathbb{P}(X \in A,\ X \neq Y) - \mathbb{P}(Y \in A,\ X \neq Y) \\
&= [\mathbb{P}(X \in A \mid X \neq Y) - \mathbb{P}(Y \in A \mid X \neq Y)] \times \mathbb{P}(X \neq Y)
\end{aligned}
$$

Coupling the 2 migrations

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$$\begin{aligned} \mathbb{P}(X \in A) - \mathbb{P}(Y \in A) &= \mathbb{P}(X = Y \in A,\ X = Y) + \mathbb{P}(X \in A,\ X \neq Y) \\ &\quad - \mathbb{P}(Y = X \in A,\ Y = X) - \mathbb{P}(Y \in A,\ X \neq Y) \\ &= \mathbb{P}(X \in A,\ X \neq Y) - \mathbb{P}(Y \in A,\ X \neq Y) \\ &= [\mathbb{P}(X \in A \mid X \neq Y) - \mathbb{P}(Y \in A \mid X \neq Y)] \times \mathbb{P}(X \neq Y) \end{aligned}$$

$$\Longrightarrow \quad \|\mathrm{Law}(X) - \mathrm{Law}(Y)\|_{tv} := \sup_A |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \leq \mathbb{P}(X \neq Y)$$

Coupling 2 migrations

$$(X_n, X'_n) = (\mathbf{i}, \mathbf{i}') \rightsquigarrow (X_{n+1}, X'_{n+1}) = (\mathbf{j}, \mathbf{j}')$$

recalling that

$$\begin{aligned} M(\mathbf{i}, \mathbf{j}) &= (1-\epsilon)\ M_\epsilon(\mathbf{i}, \mathbf{j}) + \epsilon\ \lambda(\mathbf{j}) \\ M(\mathbf{i}', \mathbf{j}') &= (1-\epsilon)\ M_\epsilon(\mathbf{i}', \mathbf{j}') + \epsilon\ \lambda(\mathbf{j}') \end{aligned}$$

KEY ϵ-coupling transition

$$\boldsymbol{M}((\mathbf{i}, \mathbf{i}'), (\mathbf{j}, \mathbf{j}')) \quad := \quad (1-\epsilon)\ M_\epsilon(\mathbf{i}, \mathbf{j}) M_\epsilon(\mathbf{i}', \mathbf{j}') + \epsilon\ \lambda(j)\ 1_{\mathbf{j}=\mathbf{j}'}$$

$\Leftrightarrow$ God flips ϵ-Head coin to define their joint evolution!

Proof: Integration the evolution of X'_n we have

$$\sum_{\mathbf{j}'} \boldsymbol{M}((\mathbf{i}, \mathbf{i}'), (\mathbf{j}, \mathbf{j}')) = M(\mathbf{i}, \mathbf{j}) \quad \text{and vice-versa}$$

$$\mathbb{P}(X_n \neq X'_n) \leq \mathbb{P}(\text{Never Head in } n \text{ trials}) = (1-\epsilon)^n$$

Birth and Death processes

population at time n after migration

$\xrightarrow{\text{branching}}$ **population at time n after birth and death**

$\xrightarrow{(n+1)\text{-th migration}}$ population at time $(n+1)$ after migration

$$\overbrace{\{I^1_{\mathbf{i},\mathbf{n}}, I^2_{\mathbf{i},\mathbf{n}}, I^3_{\mathbf{i},\mathbf{n}}, \ldots, I^{m_{\mathbf{n}}(i)}_{\mathbf{i},\mathbf{n}}\}}^{\text{individuals}} = \text{Country } c_{\mathbf{i}} \text{ at time } n \text{ with pop. } m_{\mathbf{n}}(i)$$

$$I^k_{i,n} \rightsquigarrow N^k_{i,n} \text{ offsprings } \left(I^{k,1}_{i,n}, I^{k,2}_{i,n}, \ldots, I^{k,N^k_{i,n}}_{i,n}\right)$$

with branching rates depending on country c_i attraction at time n

$$\mathbb{E}(N^k_{i,n}) \quad = \quad G_n(i)$$ **Simulation?**

Birth and Death processes

population at time n after migration

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$$\mathbb{E}(N^k_{i,n}) \quad = \quad G_n(i)$$ **Simulation?** Blackboard

Birth and Death processes $N_n = \sum_{1 \leq \mathbf{i} \leq 193} m_{\mathbf{n}}(\mathbf{i})$ random !

$$m_{(\mathbf{n}+\mathbf{1})}(\mathbf{j}) = \sum_{1\leq \mathbf{i} \leq 193} \sum_{1 \leq \mathbf{k} \leq m_{\mathbf{n}}(i)} \sum_{1 \leq \mathbf{l} \leq N^{\mathbf{k}}_{i,n}} 1_{\mathbf{j}}\left(I^{\mathbf{k},\mathbf{l}}_{\mathbf{i},\mathbf{n}}\right)$$

$$= \text{Sum of all } \mathbf{l}\text{-children of } \mathbf{k}\text{-migrants } \mathbf{i} \rightsquigarrow \mathbf{j}$$

$$\Downarrow \ \mathbb{E}(.)$$

$$q_{(\mathbf{n}+\mathbf{1})}(\mathbf{j}) = \sum_{1 \leq i \leq 193} q_{\mathbf{n}}(i) \ G_n(i) \ M_{n+1}(i, \mathbf{j})$$

$$\mathbb{E}(N_n) = \sum_{\mathbf{j}} q_{\mathbf{n}}(\mathbf{j})$$

$$= \mathbb{E}(N_{n-1}) \times \sum_i \frac{q_{\mathbf{n}}(i)}{\sum_j q_{\mathbf{n}}(j)} \ G_n(i)$$

World pop. size evolution?

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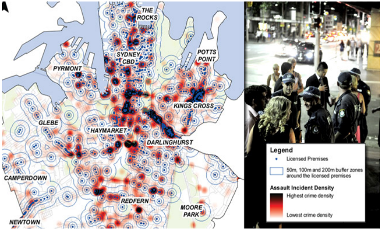
$$
\begin{aligned}
m_{(\mathbf{n+1})}(\mathbf{j}) &= \sum_{1 \leq \mathbf{i} \leq 193} \; \sum_{1 \leq \mathbf{k} \leq m_{\mathbf{n}}(i)} \; \sum_{1 \leq \mathbf{l} \leq N^{\mathbf{k}}_{i,n}} 1_{\mathbf{j}}\left(I^{\mathbf{k},\mathbf{l}}_{\mathbf{i},\mathbf{n}}\right) \\
&= \text{Sum of all } \mathbf{l}\text{-children of } \mathbf{k}\text{-migrants } \mathbf{i} \rightsquigarrow \mathbf{j} \\
& \qquad \Downarrow \; \mathbb{E}(.) \\
q_{(\mathbf{n+1})}(\mathbf{j}) &= \sum_{1 \leq i \leq 193} q_{\mathbf{n}}(i) \; G_n(i) \; M_{n+1}(i, \mathbf{j}) \\
\mathbb{E}(N_n) &= \sum_{\mathbf{j}} q_{\mathbf{n}}(\mathbf{j}) \\
&= \mathbb{E}(N_{n-1}) \times \sum_i \frac{q_{\mathbf{n}}(i)}{\sum_j q_{\mathbf{n}}(j)} \; G_n(i)
\end{aligned}
$$

World pop. size evolution?

$\rightsquigarrow$ **Super and Sub Critical!!** `Worldometer check` Blackboard

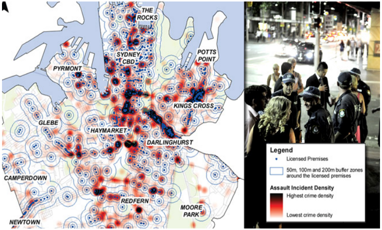
$\rightsquigarrow$ **Worlfram - Mathworld**

The traps of reinforcement



- Reinforcement $\rightsquigarrow$ make more frequent "positive" events.
- $\subset$ Learning process, natural behavior, reward-based algo, ...
- All events are related to the past, the experience,...

The traps of reinforcement



- Reinforcement $\rightsquigarrow$ make more frequent "positive" events.
- $\subset$ Learning process, natural behavior, reward-based algo, ...
- All events are related to the past, the experience,...

$$\downarrow$$

A (real) story:

- French tourist visit ever night one of the `2100 hotel pubs, taverns and bars` in Sydney
- He is attracted by pubs visited in the past.

The traps of reinforcement

- What is the stochastic model?
- How to simulate it?
- Is there some math. formulae?

The traps of reinforcement - Stochastic model

Ingredients:

- Uniform r.v. U_n on $\{1, \ldots, d\}$, with $d = 2100$ pubs.
- A "coin" with Head probability ϵ = Reinforcement rate.
- X_n = Pub visited the n-th evening

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$$\Downarrow$$

Self-reinforced model:

Given the pubs $X_0, X_2, \ldots, X_{n-1}$ visited at time $(n-1)$

$$X_n \sim \epsilon \; \frac{1}{n} \sum_{0 \leq p < n} \delta_{X_p} + (1-\epsilon) \; U_n$$

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Simulation & Analysis?

The traps of reinforcement - Stochastic model

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Simulation & Analysis?

Blackboard

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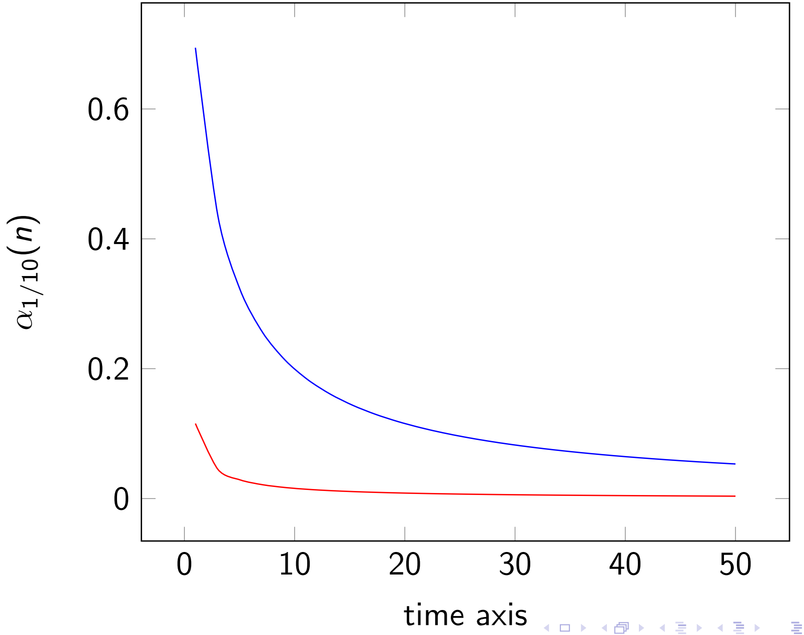
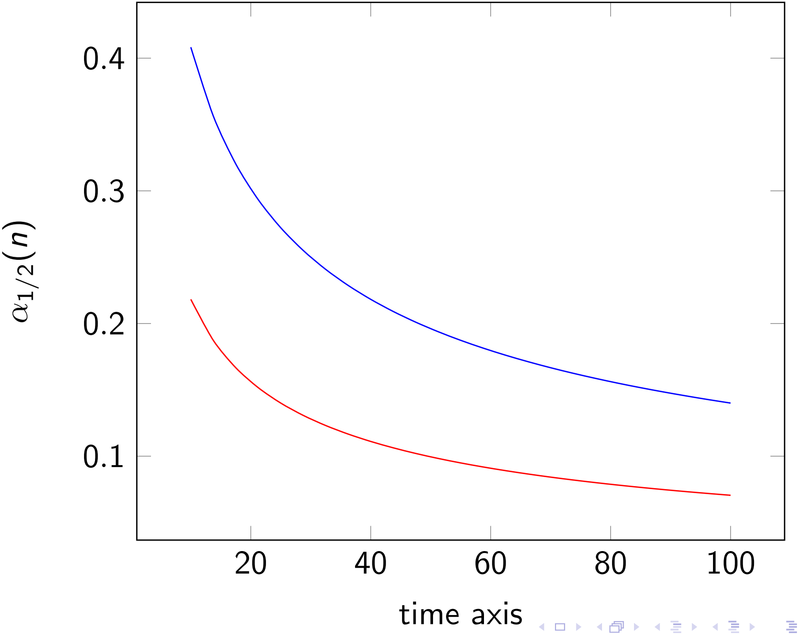
Simulation & Analysis? Blackboard

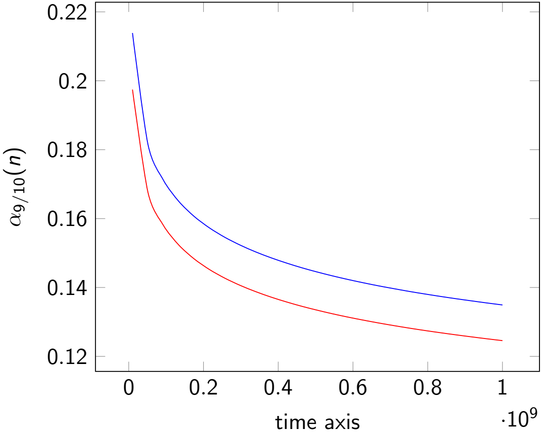
The traps of reinforcement - $\epsilon = 10\%$



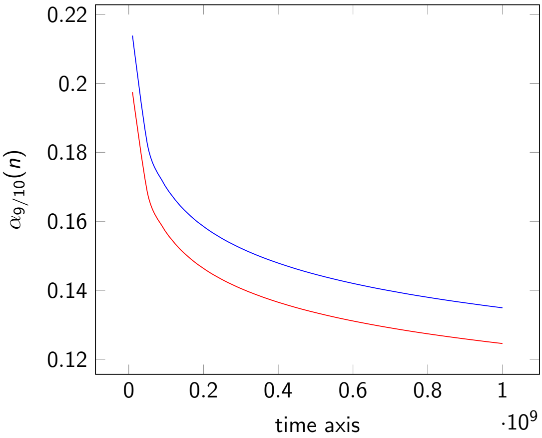
The traps of reinforcement - $\epsilon = 50\%$



The traps of reinforcement - $\epsilon = 90\%$



The traps of reinforcement - $\epsilon = 90\%$



Conclusion of the day by Henry David Thoreau (1817-1862)

Never look back unless you are planning to go that way.

Casino roulette - Double or Nothing

- Asheyl Revell (after "some" beers in a London pub) $\rightsquigarrow$ Double or Nothing in Vegas.

Casino roulette - Double or Nothing

- `Asheyl Revell` (`after "some" beers in a London pub`)
  $\rightsquigarrow$ `Double or Nothing in Vegas.`
- **Chances to win on the red color ($\mathbf{18} + \mathbf{18} = 36$)?**

Casino roulette - Double or Nothing

- `Asheyl Revell` (`after "some" beers in a London pub`) $\leadsto$ `Double or Nothing in Vegas.`
- **Chances to win on the red color** $(\mathbf{18} + \mathbf{18} = 36)$**?**

$$\Downarrow$$

$$\boxed{\mathrm{US} = 18/(36+2) = 0.474 < \mathrm{CEE} = 18/(36+1) = 0.486 \; < \; 0.5}$$

Casino roulette - Predictions?

- Starting with $\$1 \leq x < \$100$:
  Chance to win \$100 before ruin?
- How long it takes?

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⇝ **Worlfram - Mathworld**

Casino roulette - Predictions?

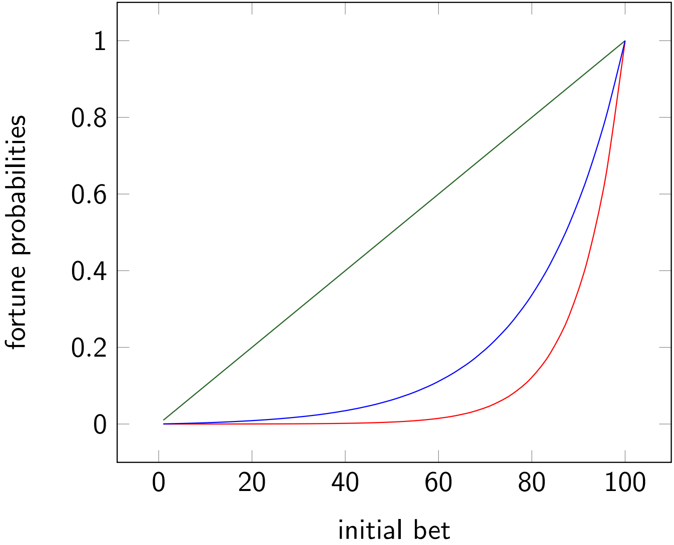
- Starting with $\$1 \leq x < \$100$:
  Chance to win \$100 before ruin?
- How long it takes?

$\rightsquigarrow$ **Worlfram - Mathworld**

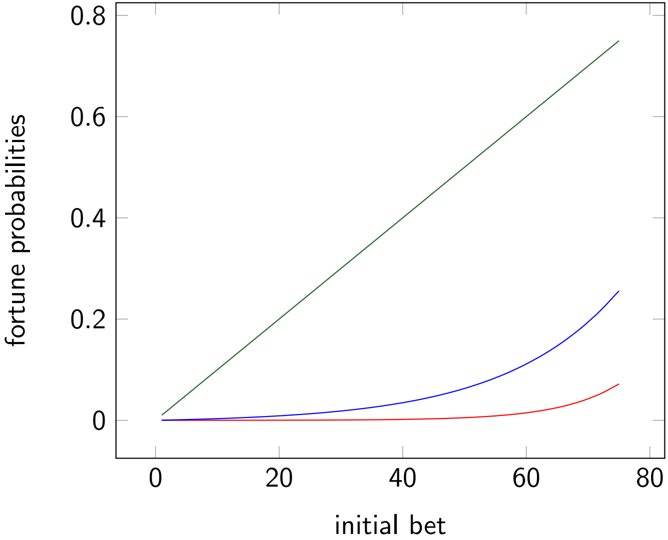
$\oplus$ Martingales betting systems = Project N^o 5

- St.Petersburg martingales
- The Grand Martingale
- The d'Alembert Martingale
- The Whittacker Martingale

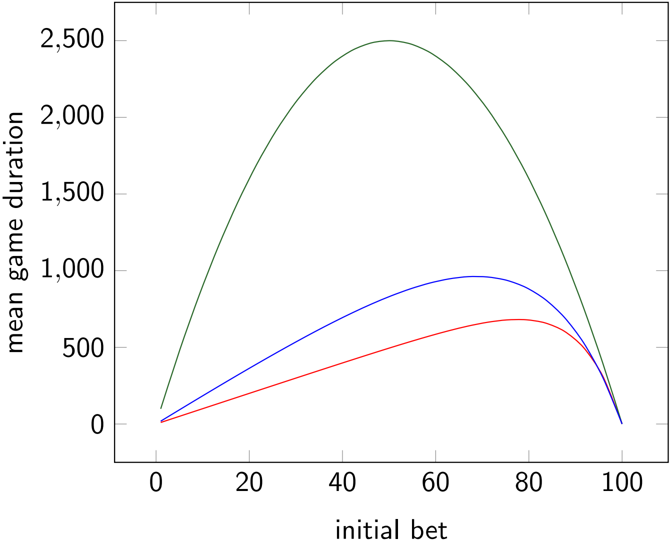
Casino roulette - some predictions



Casino roulette - some predictions



Casino roulette - some predictions



Proofs $\subset$ Martingale theory

A gambling model = Random walk!

$$Y_n = Y_0 + X_1 + \ldots + X_n \;\Leftrightarrow\; \Delta Y_n = Y_n - Y_{n-1} = X_n$$

with some initial fortune $Y_0 = y_0$ & $\perp$ bettor's profits per unit of time

$$\mathbb{P}(X_n = +1) = p \quad \text{and} \quad \mathbb{P}(X_n = -1) = q = 1 - p \;\; \in]0,1[$$

Information at time n encoded in $\mathcal{F}_n = \sigma\left(X_1, \ldots, X_n\right)$

$$\left.\begin{array}{l} \mathbb{E}\left(\Delta Y_n \mid \mathcal{F}_{n-1}\right) \\ = \mathbb{E}(X_n) \\ = p - q = \rho \end{array}\right\} = \left\{\begin{array}{cccccc} 0 & \text{when} & p = 1/2 = q & \Leftrightarrow & \text{martingale} \\ > 0 & \text{when} & p > q & \Leftrightarrow & \text{sub-martingale} \\ < 0 & \text{when} & p < q & \Leftrightarrow & \text{super-martingale} \end{array}\right.$$

Some martingale properties

$$\forall \textbf{r.v.} \qquad M_n = M_0 + \Delta M_1 + \ldots + \Delta M_n \quad \text{with} \quad \Delta M_n = M_n - M_{n-1}$$

Martingale w.r.t. some filtration of the information

$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n) \quad \text{with} \quad M_n = \varphi_n(X_0, \ldots, X_n)$$

$$\Updownarrow$$

$$\mathbb{E}(\Delta M_n \mid \mathcal{F}_{n-1}) = 0 \quad \Rightarrow \quad \mathbb{E}(M_n \mid \mathcal{F}_{n-1}) = M_{n-1} + \mathbb{E}(\Delta M_n \mid \mathcal{F}_{n-1}) = M_{n-1}$$

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$$\Rightarrow \quad \mathbb{E}(M_n \mid \mathcal{F}_{n-2}) = \mathbb{E}\left(\underbrace{\mathbb{E}(M_n \mid \mathcal{F}_{n-1})}_{=M_{n-1}} \mid \mathcal{F}_{n-2}\right) = M_{n-2}$$

Some martingale properties

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\end{aligned}$$

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$$\forall \textbf{r.v.} \qquad M_n = M_0 + \Delta M_1 + \ldots + \Delta M_n \quad \text{with} \quad \Delta M_n = M_n - M_{n-1}$$

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$$\mathcal{F}_n = \sigma(X_0, \ldots, X_n) \quad \text{with} \quad M_n = \varphi_n(X_0, \ldots, X_n)$$

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$$\begin{aligned}
\mathbb{E}(\Delta M_n \mid \mathcal{F}_{n-1}) = 0 \quad &\Rightarrow \quad \mathbb{E}(M_n \mid \mathcal{F}_{n-1}) = M_{n-1} + \mathbb{E}(\Delta M_n \mid \mathcal{F}_{n-1}) = M_{n-1} \\
&\Rightarrow \quad \mathbb{E}(M_n \mid \mathcal{F}_{n-2}) = \mathbb{E}\left(\underbrace{\mathbb{E}(M_n \mid \mathcal{F}_{n-1})}_{=M_{n-1}} \mid \mathcal{F}_{n-2}\right) = M_{n-2} \\
&\Rightarrow \quad \ldots \\
&\Rightarrow \quad \mathbb{E}(M_n \mid \mathcal{F}_p) \overset{p<n}{=} M_p \Rightarrow \mathbb{E}(M_n) = \mathbb{E}(M_0)
\end{aligned}$$

Some martingale properties

∀r.v. $\quad M_n = M_0 + \Delta M_1 + \ldots + \Delta M_n \quad \text{with} \quad \Delta M_n = M_n - M_{n-1}$

Martingale w.r.t. some filtration of the information

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$$\Rightarrow \quad \ldots$$

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$\Downarrow$ **Theo (Doob's optional stopping)**

$\mathbb{E}(M_T) = \mathbb{E}(M_0)$ **For regular "stopping times" T**

Fair game martingales (1/2)

- **Martingale Y_n**

$$\Delta Y_n = X_n \quad \text{with} \quad \mathbb{P}(X_n = +1) = \mathbb{P}(X_n = -1) = 1/2$$

- **Martingale $Z_n = Y_n^2 - n$**

$$\begin{aligned} \blacktriangledown \quad [\mathbf{Y_n^2 - n}] &= (Y_{n-1} + \Delta Y_n)^2 - \mathbf{n} \\ &= [\mathbf{Y_{n-1}^2} - (\mathbf{n-1})] + 2Y_{n-1}\Delta Y_n + (\Delta Y_n)^2 - \mathbf{1} \\ \Rightarrow \Delta Z_n &= 2Y_{n-1}\,\Delta Y_n + X_n^2 - 1 \Rightarrow \mathbb{E}(\Delta Z_n \mid \mathcal{F}_{n-1}) = 0 \quad \blacktriangle \end{aligned}$$

- **Stopping time**

  $T_{a,b}$ = first time Y_n hits the boundaries $[a,b] \overset{\text{ex.}}{=} [0,100] \ni Y_0$

Fair game martingales (2/2)

- **Martingale Y_n**

$$y_0 = \mathbb{E}(Y_{T_{a,b}}) = b\ \mathbb{P}(Y_{T_{a,b}} = b) + a\ \left(1 - \mathbb{P}(Y_{T_{a,b}} = b)\right)$$

$$\Downarrow$$

$$\mathbb{P}(Y_{T_{a,b}} = b) = (y_0 - a)/(b - a)$$

- **Martingale $Z_n = Y_n^2 - n$**

$$\begin{aligned} y_0^2 - 0 &= \mathbb{E}\left(Y_{T_{a,b}}^2\right) - \mathbb{E}(T_{a,b}) \\ &= b^2\ \mathbb{P}(Y_{T_{a,b}} = b) + a^2\ \left(1 - \mathbb{P}(Y_{T_{a,b}} = b)\right) - \mathbb{E}(T_{a,b}) \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \mathbb{E}(T_{a,b}) &= b^2\ \frac{y_0 - a}{b - a} + a^2\ \frac{b - y_0}{b - a} - y_0^2 \\ &= \dots \\ &= (b - y_0)(y_0 - a) \end{aligned}$$

Unfair game martingales (1/2)

- **Martingale** $\widetilde{Y}_n = Y_n - (p-q)\ n$

  ▼ $\Delta\widetilde{Y}_n = X_n - \mathbb{E}(X_n) = X_n - (p-q)$ ▲

- **Martingale** $Z_n = (q/p)^{Y_n}$

▼ $$\begin{aligned} \mathbf{(q/p)^{Y_n}} &= (q/p)^{Y_{n-1}+\Delta Y_n} \\ &= \mathbf{(q/p)^{Y_{n-1}}}\ (q/p)^{X_n} \\ \Rightarrow \Delta Z_n &= \mathbf{(q/p)^{Y_{n-1}}}\ \left((q/p)^{X_n} - 1\right) \Rightarrow \mathbb{E}(\Delta Z_n \mid \mathcal{F}_{n-1}) = 0 \end{aligned}$$ ▲

Unfair game martingales (2/2)

- **Martingale $Z_n = (q/p)^{Y_n}$**

$$
\begin{aligned}
(q/p)^{y_0} &= \mathbb{E}\left((q/p)^{Y_{T_{a,b}}}\right) \\
&= (q/p)^b \, \mathbb{P}(Y_{T_{a,b}} = b) + (q/p)^a \, \left(1 - \mathbb{P}(Y_{T_{a,b}} = b)\right)
\end{aligned}
$$

$$
\mathbb{P}(Y_{T_{a,b}} = b) = \frac{(q/p)^{y_0} - (q/p)^a}{(q/p)^b - (q/p)^a}
$$

- **Martingale $\widetilde{Y}_n = Y_n - (p-q)\ n$**

$$
\begin{aligned}
y_0 - (p-q) \times 0 &= \mathbb{E}\left(Y_{T_{a,b}}\right) - (p-q)\, \mathbb{E}\left(T_{a,b}\right) \\
&= b\, \mathbb{P}(Y_{T_{a,b}} = b) + a\, \left(1 - \mathbb{P}(Y_{T_{a,b}} = b)\right) \\
&\qquad - (p-q)\, \mathbb{E}\left(T_{a,b}\right)
\end{aligned}
$$

$$\Downarrow$$

$$
(p-q)\, \mathbb{E}\left(T_{a,b}\right) = (b-y_0)\, \mathbb{P}(Y_{T_{a,b}} = b) + (a-y_0)\, \left(1 - \mathbb{P}(Y_{T_{a,b}} = b)\right)
$$