Stochastic Processes

MATH5835, P. Del Moral

UNSW, School of Mathematics & Statistics

Lectures Notes 1

Consultations (RC 5112):

Wednesday 3.30 pm $\rightsquigarrow$ 4.30 pm & Thursday 3.30 pm $\rightsquigarrow$ 4.30 pm

Stochastic processes

- Random dynamical system.
- Sequential simulation of random variables.

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$$\Downarrow$$

> *The generation of random numbers is too important to be left to chance.*

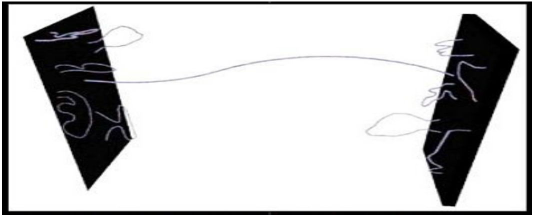
Robert Coveyou [Studies in Applied Mathematics, III (1970), 70-111.]

Lost in the Great Sloan Wall

The Tau Zero Foundation ⇝ interstellar travels in any dimension.

Tony Gonzales random travelling plans in the universe lattices at superluminal speeds

- dimension 1 and 2 from Reykjavik:



...or stretched out into massive membranes that hold whole universes

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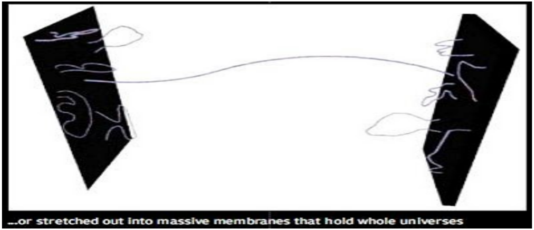

Main drawbacks: infinite returns back home....

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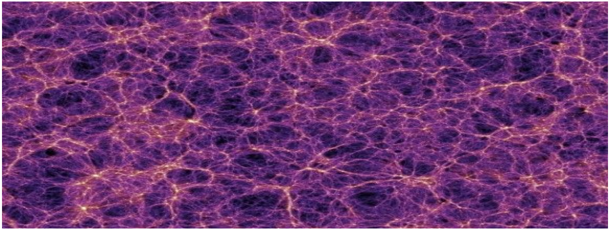
- Free trip travel voucher to the 3d- `Great Sloan Wall`:



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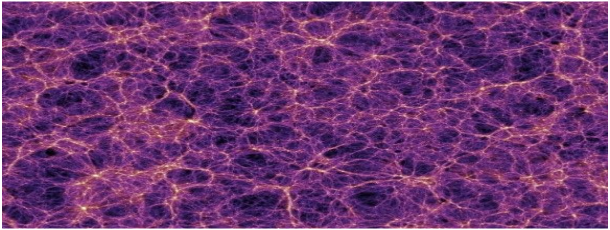

Main advantage: finite mean returns back home.

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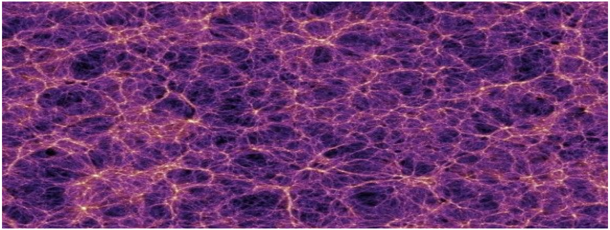

Main advantage: finite mean returns back home.

Main drawbacks:

Wanders off in the infinite universe and never returns back home!

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- Was it predictable?

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Stoch. model =Simple random walks on $\mathbb{Z}^d$:

$$d = 1 \qquad \Rightarrow X_n = X_{n-1} + U_n \quad \text{with} \quad U_n = +1 \text{ or } -1 \ \text{ proba } 1/2$$

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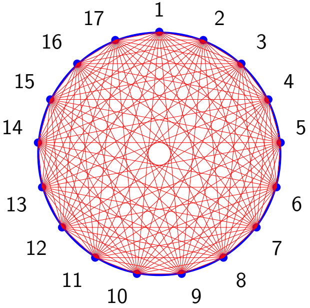
$\oplus$ More rigorous analysis for $d = 2, 3$ = Project N^o 1

Meeting Alice in Wonderland

Alice + White rabbit $\in$ Polygonal labyrinth (no communicating edges!):



- Chances to meet after 5 moves : $> 25\%$
- Chances to meet after 12 moves : $> 51\%$
- Chances to meet after 40 moves : $> 99\%$

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Neighborhood systems:

$$\text{On the set of vertices} \;\; x \sim y \Longrightarrow (x, y) \in \mathcal{E} = \text{set of edges}$$

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$$\text{Set of neighbors of } x \in \mathcal{V} := \mathcal{N}(x) = \{y \in \mathcal{V} \;:\; y \sim x\}$$

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Stochastic model:

$$X_n \text{ uniform r.v. on the set of neighbors } \mathcal{N}(X_{n-1})$$

$$\Updownarrow$$

$$\mathbb{P}\left(X_n = y \mid X_{n-1} = x\right) = \frac{1}{\#\mathcal{N}(X_{n-1})} \, 1_{\mathcal{N}(X_{n-1})}(y)$$

Meeting Alice in Wonderland - Simulation $\oplus$ Analysis

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Other application domains?

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Other application domains?

Ex.: web-graph analysis (ranking $\Rightarrow$ recommendations)

The MIT Blackjack team

Mr M. card shuffle tracking and P. Diaconis magic number

$$\leq 5 \textit{ shuffles} \Longrightarrow \textit{possible predictions} \geq 90\%,$$

$$6 \textit{ shuffles} \Longrightarrow \textit{possible predictions} \geq 40\% \ !$$

Some questions?

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$$\begin{aligned} 52! &\simeq 8.053\ 10^{67} \\ 78! &\simeq 1.13\ 10^{115} >> \text{number } 10^{80} \text{of particles in the universe} \end{aligned}$$

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Random walks on the symmetric group $\mathcal{G}_d$ with $d = 52, 78, \ldots$

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$$\begin{aligned} X_n(i) &= \text{Value of the } i\text{-th card at time } n \\ Y_n(i) &= X_n^{-1}(i) = \text{Position of the card with label } i \text{ at time } n \end{aligned}$$

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Change of order σ_{n+1} or change of values

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\begin{aligned}
i \rightsquigarrow Y_n(i) = \text{position} \quad &\rightsquigarrow \quad \sigma_{n+1}\left(Y_n(i)\right) = Y_{n+1}(i) = \text{new position} \\
i \rightsquigarrow X_n(i) = \text{value} \quad &\rightsquigarrow \quad X_n\left(\sigma_{n+1}^{-1}(i)\right) = X_{n+1}(i) = \text{new value}
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$$\Downarrow$$

$$X_n = X_{n-1} \circ \sigma_n^{-1} \quad \text{or} \quad Y_n = \sigma_n \circ Y_{n-1}$$

with some i.i.d. r.v. σ_n in some class : transpositions, top-in, riffles,. . .

Shuffling cards – Simulation+Analysis

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$$X_n \quad \text{and} \quad Y_n \longrightarrow_{n\to\infty} \text{limiting r.v.} \quad X_\infty \quad \text{and} \quad Y_\infty$$

with for any uniform r.v. σ on $\mathcal{G}_d$

$$X_\infty \overset{\text{in law}}{=} X_\infty \circ \sigma^{-1} \quad \text{and} \quad Y_\infty \overset{\text{in law}}{=} \sigma \circ Y_\infty$$

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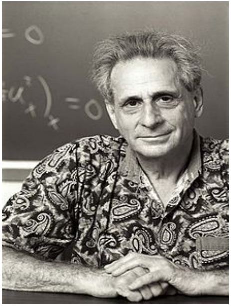
$\oplus$ Top-in & Transpositions = Project N^o 2

Kruskal count (with the classroom!)

Kruskal count - Stochastic model ⊕ Simulation ⊕ Analysis



Martin David Kruskal (1925-2006)

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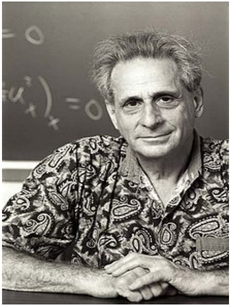
  $\rightsquigarrow$

Kruskal count - Stochastic model $\oplus$ Simulation $\oplus$ Analysis



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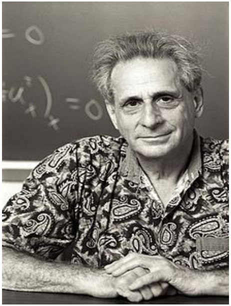

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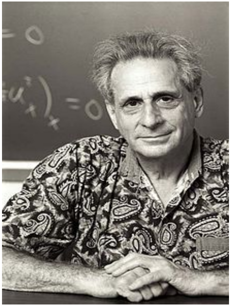

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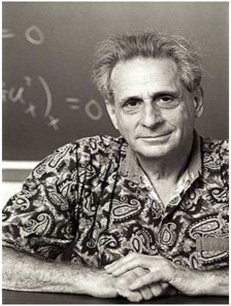

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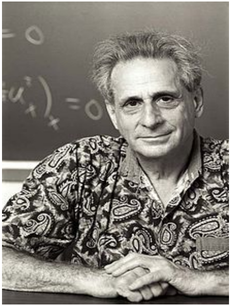

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Stoch. model = Iterated random functions (IRF):

- Some functions $x \in \mathbb{R}^2 \mapsto f_i(x) = A_i\ x + b_i \in \mathbb{R}^2$, with $i \in I = \{1, \ldots, d\}$.
- Some i.i.d. r.v. ϵ_n with some law μ on $\{1, \ldots, d\}$

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$$X_n = f_{\epsilon_n}(X_{n-1})$$

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$$\Downarrow$$

$$\boxed{X_n = f_{\epsilon_n}(X_{n-1}) = \left(f_{\epsilon_n} \circ f_{\epsilon_{n-1}} \circ \ldots \circ f_{\epsilon_1}\right)(x_0)}$$

The magic fern - The stochastic model

Stoch. model = Iterated random functions (IRF):

- Some functions $x \in \mathbb{R}^2 \mapsto f_i(x) = A_i\ x + b_i \in \mathbb{R}^2$, with $i \in I = \{1, \ldots, d\}$.
- Some i.i.d. r.v. ϵ_n with some law μ on $\{1, \ldots, d\}$

$$\Downarrow$$

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Do you believe this?

$\oplus$ Fractal & IRF = Project N^o 3

The magic fern from Daisetsuzan - An illustration

Scilab program: fractal.tree.sce

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \qquad b_1 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} r\ \cos(\varphi) & -r\ \sin(\varphi) \\ r\ \sin(\varphi) & r\ \cos(\varphi) \end{pmatrix} \qquad b_2 = \begin{pmatrix} \frac{1}{2} - \frac{r}{2}\cos(\varphi) \\ c - \frac{r}{2}\sin(\varphi) \end{pmatrix}$$

et

$$A_3 = \begin{pmatrix} q\ \cos(\psi) & -r\ \sin(\psi) \\ q\ \sin(\psi) & r\ \cos(\psi) \end{pmatrix} \qquad b_3 = \begin{pmatrix} \frac{1}{2} - \frac{q}{2}\cos(\psi) \\ \frac{3c}{5} - \frac{q}{2}\sin(\psi) \end{pmatrix}$$

with ϵ_n i.i.d. uniform on $\{1, 2, 3\}$, and with the parameters

$$\begin{aligned} c &= 0.255, \qquad r = 0.75, \qquad q = 0.625 \\ \varphi &= -\frac{\pi}{8}, \qquad \psi = \frac{\pi}{5}, \qquad |X_0| \leq 1. \end{aligned}$$

The Kepler-22b Eve

Migration in 2457 of selected 1000 humans:

Reproduction rate (20 years)

- 5.5 thousands years $\Rightarrow$ 25% population $\in$ same family.
- 6.5 thousands years $\Rightarrow$ 68% population $\in$ same family.
- 10 thousands years $\Rightarrow$ more than 99% population $\in$ same family.

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The Seven Daughters of Eve of Bryan Sykes:

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The Seven Daughters of Eve of Bryan Sykes: **Mitochondrial Eve**
$\simeq 140 - 200$ thousands years (missprint in manuscript)

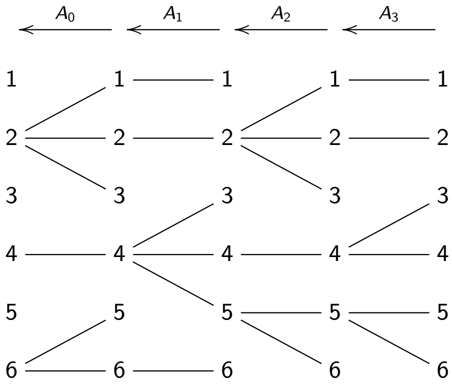
The Kepler-22b Eve - Stochastic model

Random walks on functions $a : i \in \{1, \ldots, d\} \mapsto \{1, \ldots, d\}$

The Kepler-22b Eve - Stochastic model

Random walks on functions $a : i \in \{1, \ldots, d\} \mapsto \{1, \ldots, d\}$

Birth and death = selection of the parents/ancestors/types



The Kepler-22b Eve - Simulation $\oplus$ Analysis

- **Simulation?**

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Intuitively:

$$X_n \longrightarrow_{n \to \infty} X_\infty = \text{Constant random function}$$

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$\oplus$ More rigorous analysis = Project N^o 4 

Poisson typos (reminder of stats course)

- Arrival times T_n of events :

  misprints in a text, trades counts, bus arrivals, machine failures, catastrophies, number of tries in rugby games, ...

- Stat. model on the time axis = i.i.d. exponential inter-arrivals

$$\mathbb{P}\left((T_{n+1} - T_n) \in dt \mid T_n \right) = \ e^{- \ t} \, 1_{[0,\infty[}(t) \ dt$$

- **Consequences (admitted, cf. pb.1 p. 71)**

$$\forall n \geq 0 \qquad \mathbb{P}\left(\#\{T_i \in [0,\lambda]\} = n\right) = e^{-\lambda}\,\frac{\lambda^n}{n!} \;=\; \text{Poisson r.v.}$$

and

$$\left(\frac{T_1}{T_{n+1}}, \frac{T_2}{T_{n+1}}, \ldots, \frac{T_n}{T_{n+1}}\right) \;=\; \text{ordered unif. stats on } [0,1] \;\perp\; T_{n+1}$$

- **Extension to any state space ($\supset$ the book of D. Poisson)?**

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- **Extension to any state space ($\supset$ the book of D. Poisson)?**

$\rightsquigarrow$ Counting processes with intensity $x \in S \mapsto f(x) \in [0,\infty[$

$$\mathcal{X} = \sum_{1 \leq i \leq N} \delta_{X^i}$$

with

- N Poisson r.v. with $\lambda = \int f(x)\, dx$.
- Given $N = n$, $X^1, \ldots, X^n$ i.i.d. with density $g(x) = f(x)/\lambda$.