Some stoch. models and methods in risk analysis

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The 8-th World Congress, Probability & Statistics, Session Rare event Monte Carlo Methods (J. Blanchet), Istanbul 2012

SOME HYPER-REFS:

- **Feynman-Kac formulae, Genealogical & Interacting Particle Systems with appl., Springer (2004)**
- **Sequential Monte Carlo Samplers JRSS B. (2006). (joint work with A. Doucet & A. Jasra)**
- **A Backward Particle Interpretation of Feynman-Kac Formulae M2AN (2010). (joint work with A. Doucet & S.S. Singh)**
- **On the concentration of interacting processes. Foundations & Trends in Machine Learning [170p.] (2012). (joint work with Peng Hu & Li Ming Wu) [+ Refs]**
- **More ref. on the website : Feynman-Kac particle models & their applications domains [+ Links]**

Basic notation $\mu \in \mathcal{P}(E)$ **proba. meas.,** $f \in \mathcal{B}(E)$ **bounded funct. on** E.

- $Q(x_1, dx_2)$ **integral operator** $x_1 \in E_1 \rightsquigarrow x_2 \in E_2$

$$
\begin{aligned}
Q(f)(x_1) &= \int Q(x_1, dx_2) f(x_2) \quad \text{and} \quad \mu(f) = \int \mu(dx_2)\ f(x_2) \\
[\mu Q](dx_2) &= \int \mu(dx_1) Q(x_1, dx_2) \qquad (\Longrightarrow [\mu Q](f) = \mu[Q(f)]\ )
\end{aligned}
$$

- **Boltzmann-Gibbs transformation** [$G \geq 0$ **and** μ **s.t.** $\mu(G) > 0$]

$$
\mu(dx) \quad \mapsto \quad \Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)
$$

(Bayes' rule/conditioning/restriction formula)

Importance sampling and optimal twisted measures

$$\mathbb{P}(X \in A) = \mathbb{P}_X(A) = 10^{-10} \rightsquigarrow \text{Find } \mathbb{P}_Y \text{ t.q. } \mathbb{P}_Y(A) = \mathbb{P}(Y \in A) \simeq 1$$

$\rightsquigarrow$ **Crude Monte Carlo sampling** Y^i **i.i.d.** $\mathbb{P}_Y$

$$\mathbb{P}_Y\left(\frac{d\mathbb{P}_X}{d\mathbb{P}_Y}\, 1_A\right) = \mathbb{P}_X(A) \simeq \mathbb{P}_X^N(A) := \frac{1}{N} \sum_{1 \leq i \leq N} \frac{d\mathbb{P}_X}{d\mathbb{P}_Y}(Y^i)\, 1_A(Y^i)$$

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Optimal twisted measure = Conditional distribution

$$\text{Variance} = 0 \Longleftrightarrow \mathbb{P}_Y = \text{Law}\,(X \mid X \in A) = \Psi_{1_A}(\mathbb{P}_X)$$

$$\Downarrow$$

Perfect or MCMC samplers =acceptance-rejection techniques

BUT

(Very often) with very small acceptance rates

Conditional distributions and Feynman-Kac models

Example : Markov chain models X_n restricted to subsets A_n

$$\mathbf{X} = (X_0, \ldots, X_n) \in \mathbf{A} = (A_0 \times \ldots \times A_n)$$

The conditional distributions

$$\mathrm{Law}\,(\mathbf{X} \mid \mathbf{X} \in \mathbf{A}) = \mathrm{Law}((X_0, \ldots, X_n) \mid X_p \in A_p, \quad p < n) = \mathbb{Q}_n$$

and

$$\mathrm{Proba}(X_p \in A_p, \quad p < n) = \mathcal{Z}_n$$

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$$\mathrm{Proba}(X_p \in A_p, \quad p < n) = \mathcal{Z}_n$$

are given by the Feynman-Kac measures

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n$$

with

$$\mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n) \quad \& \quad G_p = 1_{A_p}, \quad p < n$$

Feynman-Kac models (general $G_n(X_n)$ & $X_n \in E_n$)

Flow of n-marginals

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0 \le p < n} G_p(X_p)\right)$$

$$\Downarrow (\gamma_n(1) = \mathcal{Z}_n)$$

Nonlinear evolution equation :

$$\begin{array}{rcl} \eta_{n+1} & = & \Psi_{G_n}(\eta_n) M_{n+1} \\ \mathcal{Z}_{n+1} & = & \eta_n(G_n) \times \mathcal{Z}_n \end{array}$$

with the Markov transitions

$$M_{n+1}(x_n, dx_{n+1}) = \mathbb{P}(X_{n+1} \in dx_{n+1} \mid X_n = x_n)$$

Note : $[X_n = (X'_0, \ldots, X'_n)$ & $G_n(X_n) = G'(X'_n)] \Longrightarrow \eta_n = \mathbb{Q}'_n$

Interacting particle samplers

Nonlinear evolution equation :

$$\begin{array}{rcl} \eta_{n+1} & = & \Psi_{G_n}(\eta_n) M_{n+1} \\ \mathcal{Z}_{n+1} & = & \eta_n(G_n) \times \mathcal{Z}_n \end{array}$$

$\rightsquigarrow$ **(Sequential) Interacting Particle simulation technique**

G_n-acceptance-rejection with recycling $\oplus$ M_n-propositions

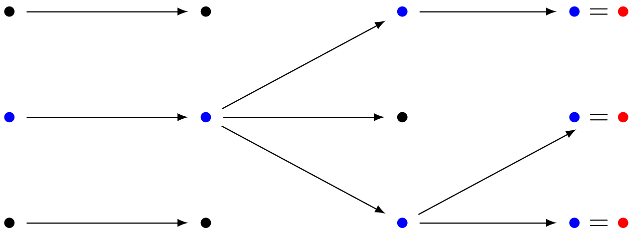
$\Updownarrow$ (**Genetic type branching/Interacting jump particle model**)

$$\xi_n = (\xi_n^i)_{1\leq i\leq N} \xrightarrow{G_n-selection} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{M_n-mutation} \xi_{n+1}$$

Key observations :

- $X_n = (X'_0, \ldots, X'_n)$, $G_n(X_n) = G'(X'_n) \Rightarrow$ **Genealogical tree model**
- Law(An ancestral line | acceptance/selection $p = 0, \ldots, n$) $= \mathbb{Q}_n$

Genealogical tree evolution $(N, n) = (3, 3)$

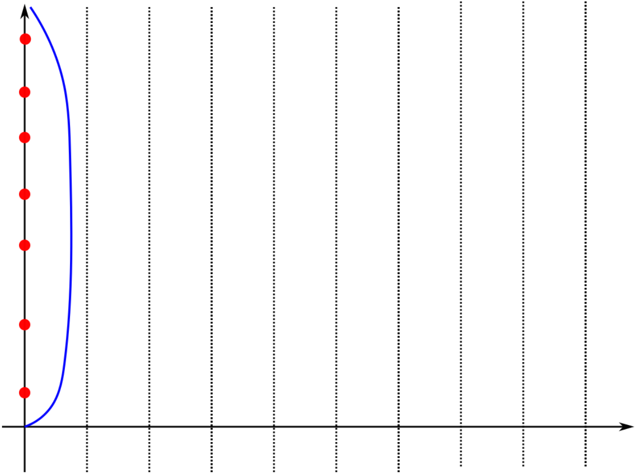


Some particle estimates

- Individuals ξ_n^i "almost" iid with law $\eta_n \simeq \eta_n^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_n^i}$
- Ancestral lines "almost" iid with law $\mathbb{Q}_n \simeq \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\mathrm{Line}_n(i)}$
- **Unbiased** Particle normalizing constants

$$\mathcal{Z}_{n+1} = \prod_{0 \leq p \leq n} \eta_p(G_p) \simeq_{N \uparrow \infty} \mathcal{Z}_{n+1}^N = \prod_{0 \leq p \leq n} \eta_p^N(G_p) \quad (\text{or } \prod \text{ success proportions})$$

Graphical illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i}$



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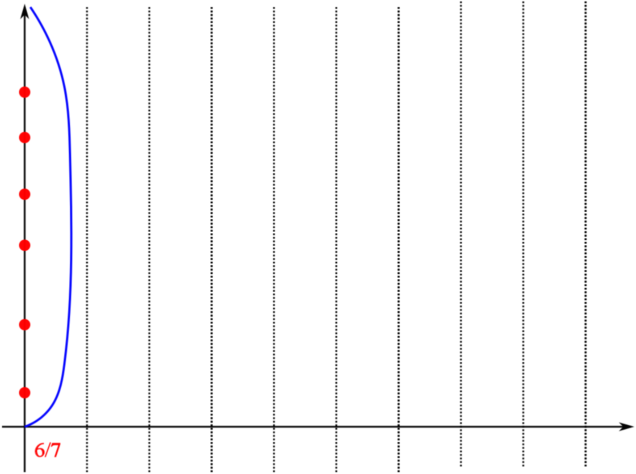

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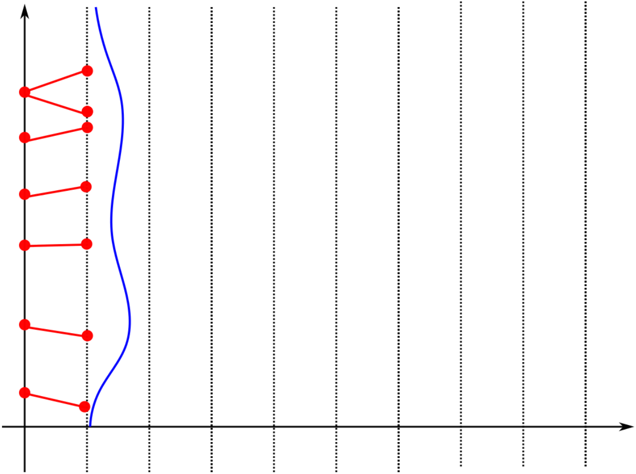

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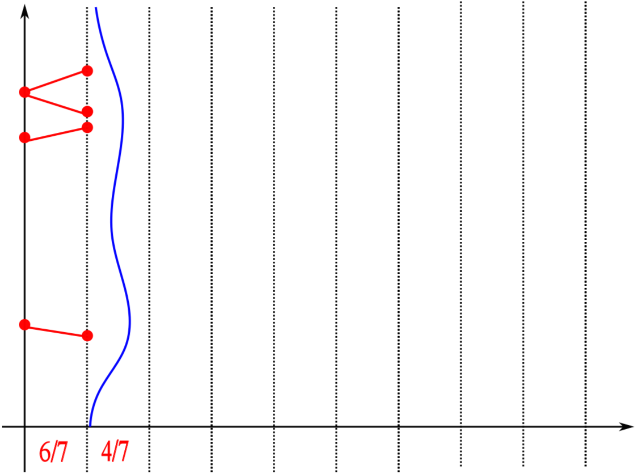

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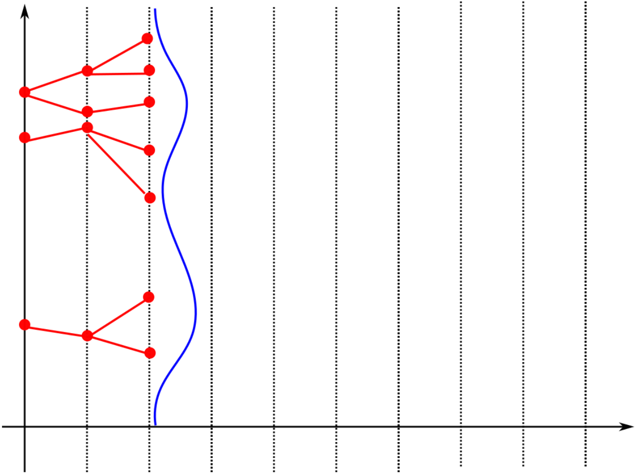

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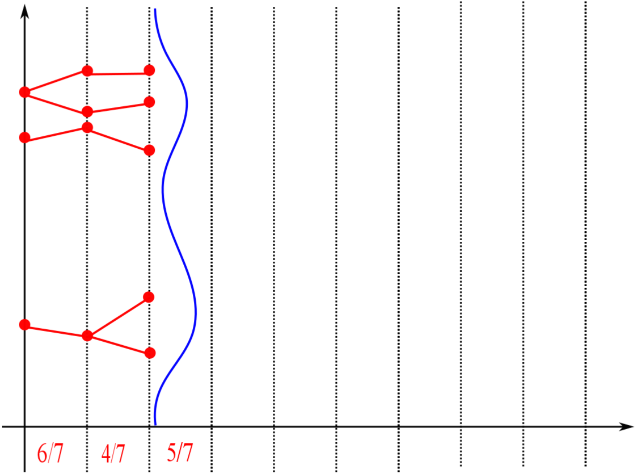

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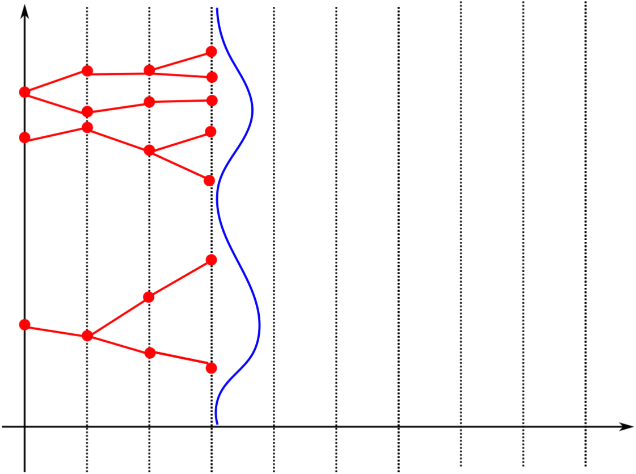

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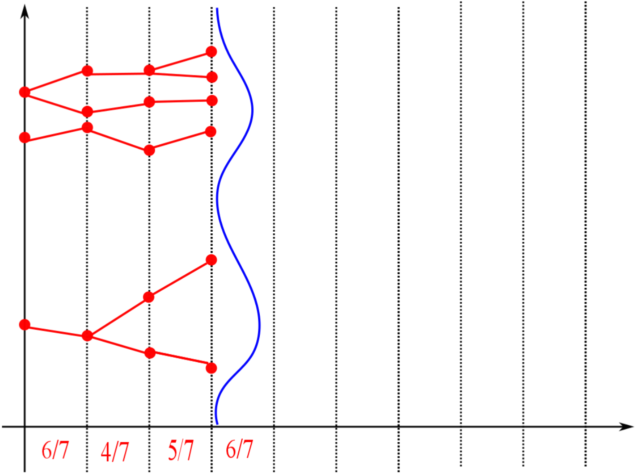

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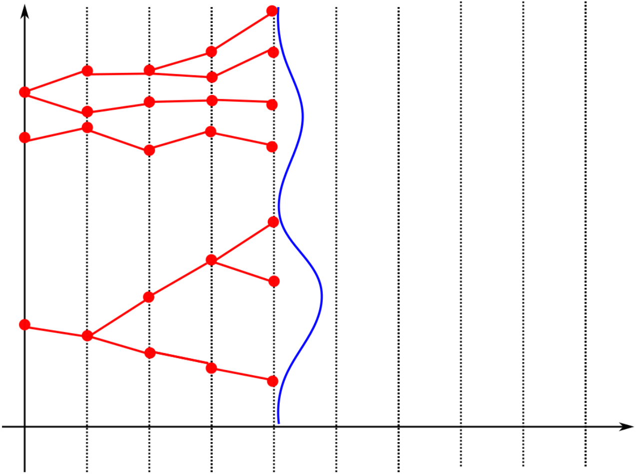

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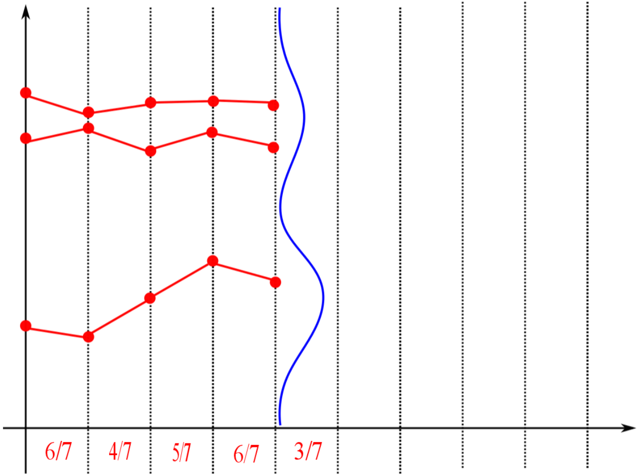

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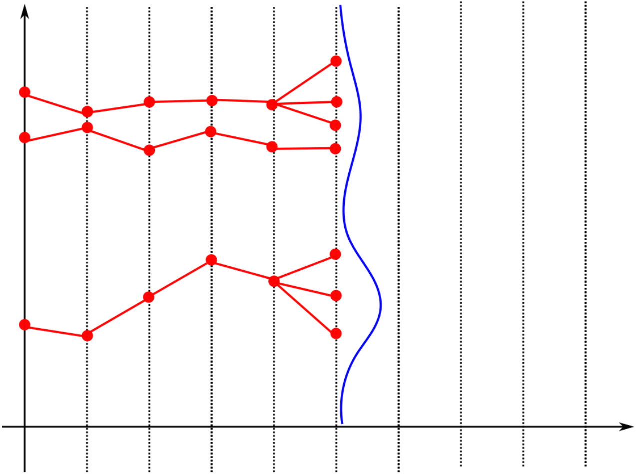

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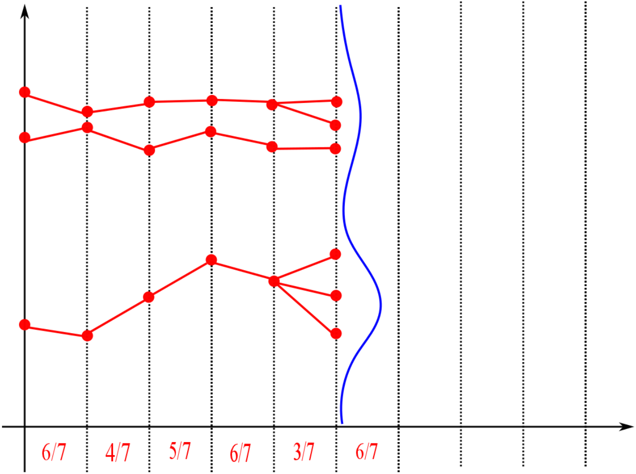

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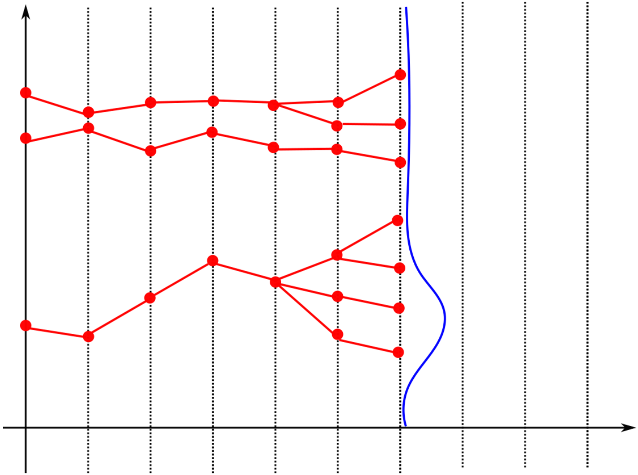

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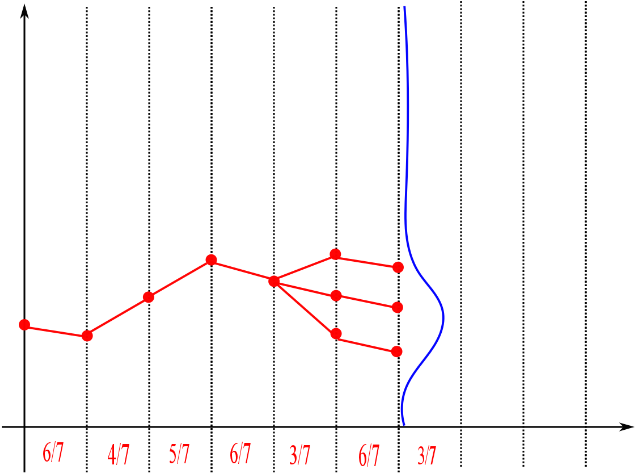

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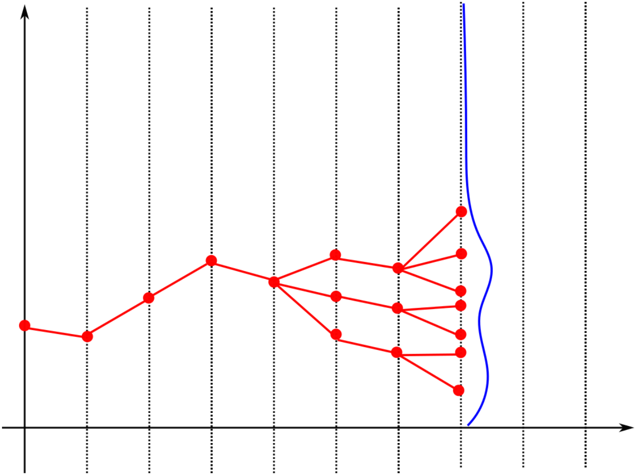

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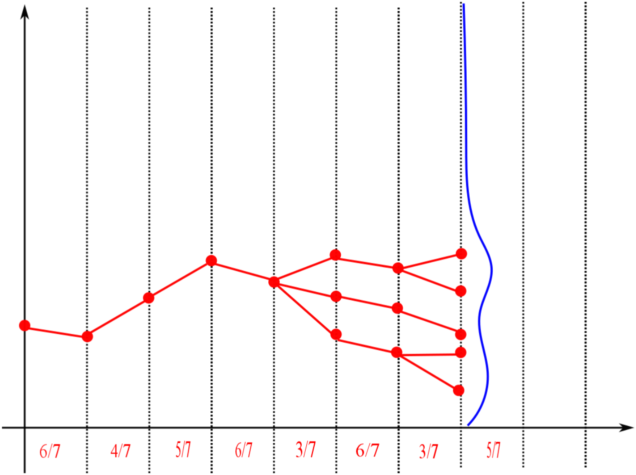

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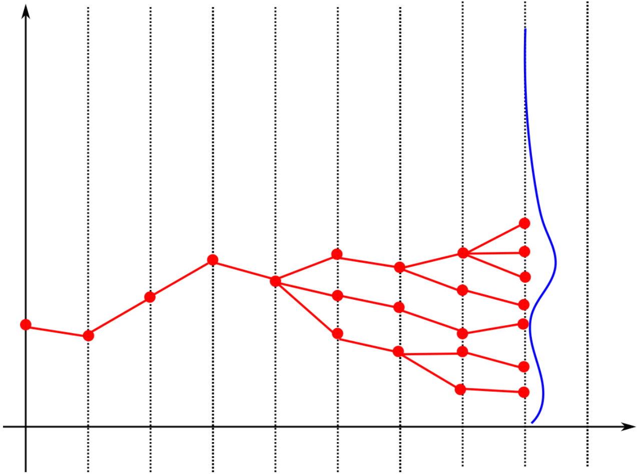

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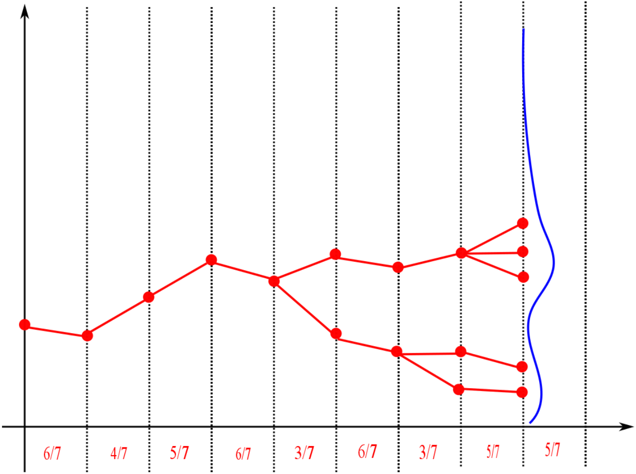

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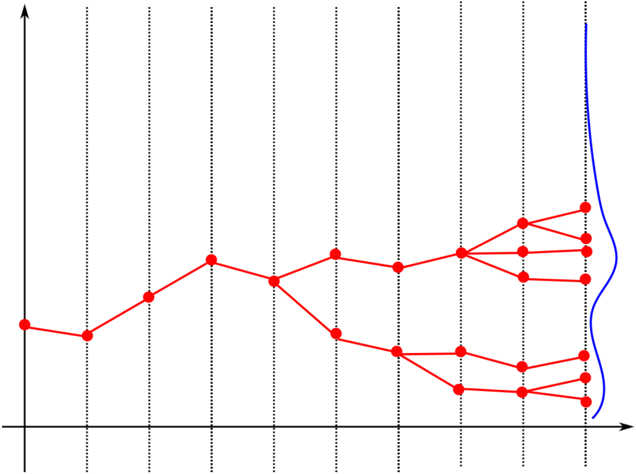

How to use the full ancestral tree model ?

$$G_{n-1}(x_{n-1})M_n(x_{n-1},dx_n) \stackrel{hyp}{=} H_n(x_{n-1},x_n)\ \nu_n(dx_n)$$

$\Rightarrow$ **Backward Markov model :**

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) = \eta_n(dx_n)\quad \underbrace{\mathbb{M}_{n,\eta_{n-1}}(x_n,dx_{n-1})}_{\propto\ \eta_{n-1}(dx_{n-1})\ H_n(x_{n-1},x_n)}\ \ldots\mathbb{M}_{1,\eta_0}(x_1,dx_0)$$

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Particle approximation

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0^N}(x_1,dx_0)$$

Ex.: Additive functionals $\mathbf{f_n}(x_0,\ldots,x_n) = \frac{1}{n+1}\sum_{0\leq p\leq n} f_p(x_p)$

$$\mathbb{Q}_n^N(\mathbf{f_n}) := \frac{1}{n+1}\sum_{0\leq p\leq n}\eta_n^N\ \underbrace{\mathbb{M}_{n,\eta_{n-1}^N}\ldots\mathbb{M}_{p+1,\eta_p^N}(f_p)}_{\text{(random) matrix operations}}$$

Self avoiding walks in $\mathbb{Z}^d$

Feynman-Kac model with

$$\mathbf{X_n} = (X_0, \ldots, X_n) \quad \& \quad G_n(\mathbf{X_n}) = 1_{X_n \notin \{X_0, \ldots, X_{n-1}\}}$$

$$\Updownarrow$$

Conditional distributions

$$\mathbb{Q}_n = \mathrm{Law}\left((\mathbf{X_0}, \ldots, \mathbf{X_n}) \mid X_p \neq X_q, \ \forall 0 \leq p < q < n\right)$$

and

$$\mathcal{Z}_n = \mathrm{Proba}\left(X_p \neq X_q, \ \forall 0 \leq p < q < n\right)$$

Level crossing probabilities (1)

$$\mathbb{P}\left(V_n(X_n) \geq a\right) \quad \text{or} \quad \mathbb{P}\left(X \text{ hits } A_n \text{ before } B\right)$$

- Level crossing at a fixed given time

$$\begin{aligned} \mathbb{P}\left(V_n(X_n) \geq a\right) &= \mathbb{E}\left(f_n(X_n)\; e^{V_n(X_n)}\right) \\ &= \mathbb{E}\left(\mathbf{f_n}(\mathbf{X}_n) \prod_{0 \leq p < n} G_p(\mathbf{X}_p)\right) \end{aligned}$$

with

- The Markov chain on transition space

$$\mathbf{X}_n = (X_n, X_{n+1}) \quad \text{and} \quad G_n(\mathbf{X}_n) = \exp\left[V_{n+1}(X_{n+1}) - V_n(X_n)\right]$$

- The test functions

$$f_n(\mathbf{X}_n) = 1_{V_n(X_n) \geq a}\; e^{-V_n(X_n)}$$

Level crossing probabilities (2)

- Excursion level crossing $A_n \downarrow$, with B non critical recurrent subset.

$$\mathbb{P}(X \text{ hits } A_n \text{ before } B) \quad = \quad \mathbb{E}\left(\prod_{0\leq p\leq n} 1_{A_p}(X_{T_p})\right)$$

$$T_n := \inf\{p \geq T_{n-1} \;:\; X_p \in (A_n \cup B)\}$$

Feynman-Kac model

$$\mathbb{E}\left(\prod_{0\leq p\leq n} 1_{A_p}(X_{T_p})\right) = \mathbb{E}\left(\prod_{0\leq p<n} G_p(\mathbf{X}_p)\right)$$

with

$$\mathbf{X}_n = (X_t)_{t\in[T_n,T_{n+1}]} \quad \& \quad G_n(\mathbf{X}_n) = 1_{A_{n+1}}(X_{T_{n+1}})$$

Absorption models (Ground state energies, extinction probabilities, trapping models)

- Sub-Markov semigroups

$$Q_n(x, dy) = G_{n-1}(x)\ M_n(x, dy) \rightsquigarrow E_n^c = E_n \cup \{c\}$$

- Absorbed Markov chain

$$X_n^c \in E_n^c \xrightarrow{\textit{absorption} \sim (1-G_n)} \widehat{X}_n^c \xrightarrow{\textit{exploration} \sim M_n} X_{n+1}^c$$

$$\Downarrow$$

$$\mathbb{Q}_n = \mathrm{Law}((X_0^c, \ldots, X_n^c) \mid T^{\textit{absorption}} \geq n)$$

and

$$\mathcal{Z}_n = \mathrm{Proba}\left(T^{\textit{absorption}} \geq n\right)$$

Homogeneous models $(G_n, M_n) = (G, M)$

- Reversibility condition : $\mu(dx)M(x, dy) = \mu(dy)M(y, dx)$

$$\mathrm{Proba}\left(T^{absorption} \geq n\right) \simeq \lambda^n$$

with $\lambda =$ top eigenvalue of

$$Q(x, dy) = G(x)\ M(x, dy)$$

- Frobenius theorem $\rightsquigarrow \exists!\ h\ s.t.\ \ Q(h) = \lambda h$
  - Quasi-invariant/Yaglom measure/F-K sg fixed point :

$$\mathbb{P}(X_n^c \in dx \mid T^{absorption} > n) \rightarrow_{n\uparrow\infty} \frac{1}{\mu(h)}\ h(x)\ \mu(dx)$$

  - Doob h-process X^h :

$$M^h(x, dy) = \frac{1}{\lambda} h^{-1}(x) Q(x, dy) h(y) = \frac{Q(x, dy)h(y)}{Q(h)(x)} = \frac{M(x, dy)h(y)}{M(h)(x)}$$

Homogeneous models $(G_n, M_n) = (G, M)$

$$\mathbb{Q}_n(d(x_0,\ldots,x_n)) \propto \mathbb{P}((X_0^h,\ldots,X_n^h) \in d(x_0,\ldots,x_n))\ h^{-1}(x_n)$$

- *Invariant measure $\mu_h = \mu_h M^h$ & normalized additive functionals*

$$\overline{F}_n(x_0,\ldots,x_n) = \frac{1}{n+1}\sum_{0\le p\le n} f(x_p) \Longrightarrow \mathbb{Q}_n(\overline{F}_n) \simeq_n \mu_h(f)$$

- If $G = G^\theta$ depends on some $\theta \in \mathbb{R} \rightsquigarrow f := \frac{\partial}{\partial\theta}\log G^\theta$

$$\frac{\partial}{\partial\theta}\log\lambda^\theta \simeq_n \frac{1}{n+1}\frac{\partial}{\partial\theta}\log\mathcal{Z}^\theta_{n+1} = \mathbb{Q}_n(\overline{F}_n)$$

NB : Similar expression when M^θ depends on some $\theta \in \mathbb{R}$.

Boltzmann-Gibbs measures (Stochastic optimization, free energies, bayesian inference)

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n}\; e^{-\beta_n V(x)}\; \lambda(dx) \quad \text{with} \quad \beta_n \uparrow$$

- For any MCMC transition M_n with target η_n

$$\eta_n = \eta_n M_n$$

- Updating of the temperature parameter

$$\eta_{n+1} = \Psi_{G_n}(\eta_n) \quad \text{with} \quad G_n = e^{-(\beta_{n+1}-\beta_n)V}$$

Proof : $\quad e^{-\beta_{n+1}V} = e^{-(\beta_{n+1}-\beta_n)V} \times e^{-\beta_n V}$

Consequence :

$$\eta_{n+1} = \eta_{n+1} M_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$$

and $(\beta_0 = 0)$

$$\lambda\left(e^{-\beta_n V}\right) = \mathcal{Z}_n = \prod_{0\leq p<n} \eta_p(G_p)$$

Restriction models (Uncertainty propagations, tail probabilities, black box input/output)

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n} 1_{A_n}(x)\ \lambda(dx) \quad \text{with} \quad A_n \downarrow$$

- For any MCMC transition M_n with target η_n

$$\eta_n = \eta_n M_n$$

- Updating of the subset

$$\eta_{n+1} = \Psi_{G_n}(\eta_n) \quad \text{with} \quad G_n = 1_{A_{n+1}}$$

Proof : $\quad 1_{A_{n+1}} = 1_{A_{n+1}} \times 1_{A_n}$

Consequence :

$$\eta_{n+1} = \eta_{n+1} M_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$$

and $(\lambda(A_0) = 1)$

$$\lambda(A_n) = \mathcal{Z}_n = \prod_{0 \le p < n} \eta_p(G_p)$$

Equivalent Algorithms

Sequential Monte Carlo	Sampling	Resampling
(Particle) Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Dynamic Population	Exploration	Branching
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$1950 \leq$ Heuristic style algo. $\leq 1996 \leq$ Particle Feynman-Kac models

Convergence analysis $\supset$ CLT, LDP, $\mathbb{L}_p$-estimates, Empirical processes convergence, Moderate deviations, Berry-Essen, Propagations of chaos, Exact weak expansions, ...

Concentration analysis = Exponential deviation proba. estimates

Current population models

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time.
Test funct. $\|f_n\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \eta_n\right](f) \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

is greater than $1 - e^{-x}$.

- $x = (x_i)_{1 \leq i \leq d} \rightsquigarrow (-\infty, x] = \prod_{i=1}^{d} (-\infty, x_i]$ cells in $E_n = \mathbb{R}^d$.

$$F_n(x) = \eta_n\left(1_{(-\infty,x]}\right) \quad \text{and} \quad F_n^N(x) = \eta_n^N\left(1_{(-\infty,x]}\right)$$

The probability of the following event

$$\sqrt{N}\ \left\|F_n^N - F_n\right\| \leq c\ \sqrt{d\ (x+1)}$$

is greater than $1 - e^{-x}$.

Particle free energy models

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time
$\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- Unbias prop. ($\rightsquigarrow$ *Island models [cf. DM, Dubarry, Moulines CS2]*)

$$\mathbb{E}\left(\eta_n^N(f_n) \prod_{0 \leq p < n} \eta_p^N(G_p)\right) = \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

- For any $\epsilon \in \{+1, -1\}$, the probability of the event

$$\frac{\epsilon}{n} \log \frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

is greater than $1 - e^{-x}$.

note $(0 \leq \epsilon \leq 1 \Rightarrow (1 - e^{-\epsilon}) \vee (e^{\epsilon} - 1) \leq 2\epsilon)$

$$e^{-\epsilon} \leq \frac{z^N}{z} \leq e^{\epsilon} \Rightarrow \left|\frac{z^N}{z} - 1\right| \leq 2\epsilon$$

Genealogical tree models $:= \eta_n^N$ (in path space)

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time
$\mathbf{f_n}$ test function $\|\mathbf{f_n}\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \mathbb{Q}_n\right](\mathbf{f_n}) \leq c_1\ \frac{n+1}{N}\ \left(1 + x + \sqrt{x}\right) + c_2\ \sqrt{\frac{(n+1)}{N}}\ \sqrt{x}$$

is greater than $1 - e^{-x}$.

Backward particle models

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time.
$\mathbf{f_n}$ normalized additive functional with $\|f_p\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- The probability of the event

$$\left[\mathbb{Q}_n^N - \mathbb{Q}_n\right](\mathbf{f}_n) \leq c_1 \ \frac{1}{N}\ (1 + (x + \sqrt{x})) + c_2\ \sqrt{\frac{x}{N(n+1)}}$$

  is greater than $1 - e^{-x}$.

- $\mathbf{f}_{a,n}$ normalized additive functional w.r.t. $f_p = 1_{(-\infty,a]}$, $a \in \mathbb{R}^d = E_n$
  .
  The probability of the following event

$$\sup_{a \in \mathbb{R}^d} \left|\mathbb{Q}_n^N(\mathbf{f_{a,n}}) - \mathbb{Q}_n(\mathbf{f_{a,n}})\right| \leq c\ \sqrt{\frac{d}{N}(x+1)}$$

  is greater than $1 - e^{-x}$.