

New Insights on Particle MCMC algorithms

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ASC-IMS 2014 Conference

Some hyper-refs

- ▶ [PMCMC - Andrieu, Doucet, Holenstein JRSS-10](#)
- ▶ [On Feynman-Kac and PMCMC models, with R. Kohn and F. Patras \(ArXiv-2014\).](#)
- ▶ [On parallel implementation of Sequential Monte Carlo methods: the island particle model, with C. Vergé, C. Dubarry, and E. Moulines. \(Statistics and Computing-2013\).](#)
- ▶ [A Backward Particle Interpretation of Feynman-Kac Formulae, with A. Doucet and S. Singh \(Arxiv-2009/M2AN-2010\)](#)
- ▶ [Feynman-Kac formulae, Springer \(2004\) \[+ Refs\]](#)
- ▶ [Mean field simulation for Monte Carlo integration. Chapman - Hall \(2013\) \[+ Refs\]](#)

Bayes & Feynman-Kac models

Origins/Equivalent particle algorithms

Ex.: MCMC with product target measures

Particle measures \oplus 2 key formulae

Conditioning and duality formulae

Taylor expansions for PMCMC transitions

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$$p(x_{0:n} \mid y_{0:n}) \propto \underbrace{p(y_{0:n} \mid x_{0:n})}_{\prod_{0 \leq k \leq n} p(y_k \mid x_k) \leftarrow \text{likelihood functions } G_k(x_k)} \times \underbrace{p(x_{0:n})}_{\rightsquigarrow \text{density of } \mathbb{P}_n}$$

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strictly \subset Feynman-Kac models ($\exists \neg !$)

$$\mathbb{Q}_n(d(x_0, \dots, x_n)) = \frac{1}{Z_n} \left\{ \prod_{0 \leq p < n} G_p(x_p) \right\} \mathbb{P}_n(d(x_0, \dots, x_n))$$

Notation : η_n n-th marginal distribution

Key obs.: $\mathbf{X}_n = (X_0, \dots, X_n)$ and $\mathbf{G}_k(\mathbf{X}_k) = G_k(X_k)$

$$\eta_n(f) = \mathbb{E} \left(f(\mathbf{X}_n) \prod_{0 \leq k < n} \mathbf{G}_k(\mathbf{X}_k) \right) = \mathbb{Q}_n(f)$$

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| \Leftrightarrow Algo. | $X_{n-1} \rightsquigarrow X_n$ | G_n |
|----------------------------|--------------------------------|-----------------------|
| Sequential Monte Carlo | Sampling | Resampling |
| Particle Filters | Prediction | Updating |
| Genetic Algorithms | Mutation | Selection |
| Evolutionary Population | Exploration | Branching-selection |
| Diffusion Monte Carlo | Free evolutions | Absorption |
| Quantum Monte Carlo | Walkers motions | Reconfiguration |
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bootstrapping, spawning, cloning, pruning, replenish, multi-level splitting, enrichment, go with the winner, quantum teleportation, . . .

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Convergence/Performance analysis : CLT, LDP, \mathbb{L}_p -estimates, Empirical processes, Moderate deviations, propagations of chaos, unif cv w.r.t. time, new stochastic models....

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\forall Interpolating path measures $\pi_0 \rightsquigarrow \dots \rightsquigarrow \pi_T$

$$\pi_n(d\theta) \propto \left\{ \prod_{1 \leq k \leq n} h_k(\theta) \right\} \lambda(d\theta)$$

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As the filtering equation:

$$\pi_{n-1} \xrightarrow{\text{Correction/Updating}} d\pi_n \propto h_n \, d\pi_{n-1} \xrightarrow{\pi_n\text{-MCMC/Prediction } M_n} \pi_n$$

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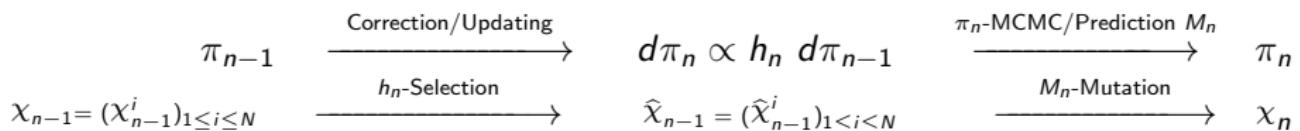
$$\begin{array}{c} \pi_{n-1} \\ \chi_{n-1} = (\chi_{n-1}^i)_{1 \leq i \leq N} \end{array} \xrightarrow{\begin{array}{c} \text{Correction/Updating} \\ \hline h_n\text{-Selection} \end{array}} \begin{array}{c} d\pi_n \propto h_n \, d\pi_{n-1} \\ \hat{\chi}_{n-1} = (\hat{\chi}_{n-1}^i)_{1 \leq i \leq N} \end{array} \xrightarrow{\begin{array}{c} \pi_n\text{-MCMC/Prediction } M_n \\ \hline \end{array}} \pi_n$$

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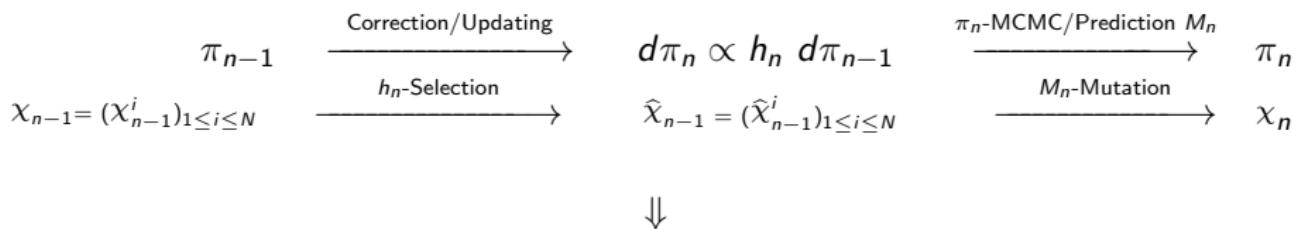


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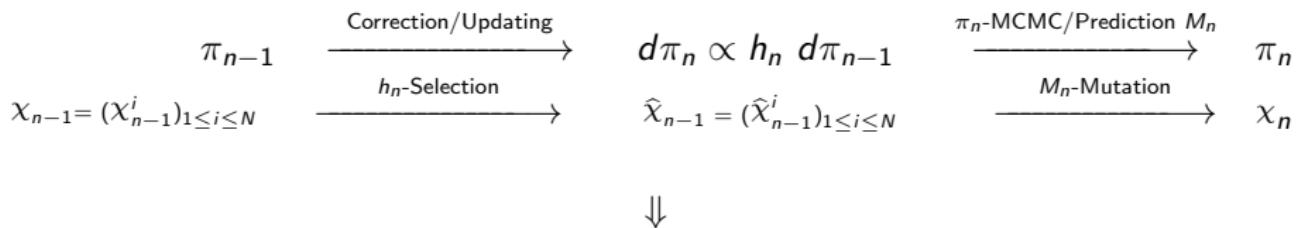
subset n-th marginals $\pi_n = \eta_n$ of a Feynman-Kac model \mathbb{Q}_n

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C n-th marginals $\pi_n = \eta_n$ of a Feynman-Kac model \mathbb{Q}_n

- Physics \rightsquigarrow Crook/Jarzinsky formula;
- Rare event \rightsquigarrow subset sampling/multi-level splitting;
- Operation Research \rightsquigarrow Interacting simulated annealing ...)

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$$\eta_n \simeq \eta_n^{\textcolor{red}{N}} := \frac{1}{N} \sum_{1 \leq i \leq N} \delta(\chi_{0,n}^i, \chi_{1,n}^i, \dots, \chi_{n,n}^i) = \text{i-th ancestral line}$$

$\rightsquigarrow \mathbb{X}_n := \text{uniform ancestral line}$

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1) Product formulae/Particle approximation

$$\mathcal{Z}_n = \prod_{0 \leq k < n} \eta_k(G_k) \stackrel{\text{unbias}}{\simeq} \prod_{0 \leq k < n} \eta_k^{\text{N}}(G_k) := \mathcal{Z}_n^{\text{N}} = \prod_{0 \leq k < n} \mathcal{G}_k(\chi_k)$$

with the empirical potential function

$$\mathcal{G}_k(\chi_k) = \eta_k^{\text{N}}(G_k) = \frac{1}{N} \sum_{1 \leq i \leq N} G_k(\chi_k^i)$$

\rightsquigarrow Many-body FK on path space

FK model with $(X_n, G_n) \rightsquigarrow (\mathcal{X}_n, \mathcal{G}_n) = \text{Many-body FK}$

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For any empirical function $F(\chi_n) = \frac{1}{N} \sum_{1 \leq i \leq N} f(\chi_n^i)$

Theo [MPRF-1996]

$$\mathbb{E} \left(f(\mathbf{X}_n) \prod_{0 \leq k < n} \mathbf{G}_k(\mathbf{X}_k) \right) = \mathbb{E} \left(F(\mathcal{X}_n) \prod_{0 \leq k < n} \mathcal{G}_k(\mathcal{X}_k) \right)$$

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- \rightsquigarrow SC-13 (Island models/Parallel particle models)
- \rightsquigarrow FTML-02/Arxiv-2011 (independent Metropolis-Hastings/SMC²)

The 2nd key

Hypothesis

$$M_{k+1}(x_k, dx_{k+1}) = H_{k+1}(x_k, x_{k+1}) \lambda(dx_{k+1}) \stackrel{ex.}{\propto} e^{-\frac{1}{2}(x_{k+1} - a(x_k))^2} dx_{k+1}$$

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2) Backward formulae/Backward Particle chain

$$\mathbb{Q}_n(d(x_0, \dots, x_n)) = \eta_n(dx_n) \mathbb{K}_{n,\eta_{n-1}}(x_n, dx_{n-1}) \dots \mathbb{K}_{1,\eta_0}(x_1, dx_0)$$

with

$$\mathbb{K}_{k+1,\eta_k}(x_{k+1}, dx_k)$$

$$= \frac{\eta_k(dx_k) G_k(x_k) H(x_k, x_{k+1})}{\int \eta_k(dx'_k) G_k(x'_k) H(x'_k, x_{k+1})}$$

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~~> backward random path \mathbb{X}_n

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Unbias Many-body FK (cf. M2AN-2010 / Arxiv-2014)

\mathbb{X}_n = Uniform ancestral line

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\oplus

Law (Ancestral line | complete tree) = Law of the backward particle model

Theo 1 [Duality formula for Many-body FK ([Arxiv-2014](#))]

$$\mathbb{E} \left(F(\mathbb{X}_n, \mathcal{X}_n) \prod_{0 \leq k < n} G_k(\mathcal{X}_k) \right) = \mathbb{E} \left(F(\mathbf{X}_n, \mathcal{X}_n) \prod_{0 \leq k < n} G_k(\mathbf{X}_k) \right)$$

with

$$\begin{aligned} \text{Law}(\mathcal{X}_n \mid \mathbf{X}_n = \mathbf{x}) &= \text{Law } N \text{ particles with frozen path } \mathbf{X}_n = \mathbf{x} \\ &= \text{Law}_{\text{many-body}}(\mathcal{X}_n \mid \mathbb{X}_n = \mathbf{x}) \end{aligned}$$

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Two Feynman-Kac - reversible transitions $\mathbb{K}_n(x, dx')$

$$x \rightsquigarrow \mathcal{X}_n \text{ with frozen path } x \rightsquigarrow \begin{cases} \text{Random backward path } x' \\ \text{or} \\ \text{Any random ancestral line } x' \end{cases}$$

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⊕ Direct (but too crude) minorization condition $\mathbb{K}_n(x, \cdot) \geq \epsilon_n \eta_n$

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$$\mathbb{K}_n(x, \cdot) = \eta_n + \sum_{1 \leq k \leq l} \frac{1}{N^k} d^{(k)} \mathbb{K}_n(x, \cdot) + O\left(\frac{1}{N^{l+1}}\right)$$

at any order l , with explicit operators $d^{(k)} \mathbb{K}_n$ in terms of coalescent trees

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Some direct corollaries

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- ▶ m -Iterates expansions

$$\mathbb{K}_n^m(x, \cdot) = \eta_n + \frac{1}{N^m} \left[\sum_{0 \leq k \leq l} \frac{1}{N^k} d^{(m+k)} \mathbb{K}_n^m(x, \cdot) + O\left(\frac{1}{N^{l+1}}\right) \right]$$