Optimal Nonlinear Filtering in GPS/INS Integration

H. CARVALHO

P. DEL MORAL
LAAS, DIGILOG
France

A. MONIN

G. SALUT
LAAS
France

The application of optimal nonlinear/non-Gaussian filtering to the problem of INS/GPS integration in critical situations is described. This approach is made possible by a new technique called particle filtering, and exhibits superior performance when compared with classical suboptimal techniques such as extended Kalman filtering. Particle filtering theory is introduced and GPS/INS integration simulation results are discussed.

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Authors' addresses: H. Carvalho, A. Monin, G. Salut, Laboratoire d'Analyse et d'Architecture des Systèmes (LAAS), CNRS, 7 Avenue du Colonel Roche, 31077 Toulouse CEDEX France. P. Del Moral is now with Laboratoire de Statistique et de Probabilite's, UPS, 118 Route de Narbonne, 31062, Toulouse, France.

I. NOMENCLATURE

α	Regularizing function
$\mathbb{E}[\cdot]$	Mathematical expectation
$\mathbb{E}[\cdot \mid \cdot]$	Conditional expectation
e_λ, e_ϕ	INS errors in longitude and latitude
INS	Inertial Navigation System
λ, ϕ	Longitude, latitude
λ_I, ϕ_I	Longitude and latitude readings from INS
λ_m, ϕ_m	Longitude and latitude of vehicle
N	Number of particles
$dP(\cdot), dP_0(\cdot)$	Probability measures
$dP(\cdot \mid \cdot)$	Conditional probability
p_k^i	Weight of particle i
ρ_i	Pseudo-range from satellite i
X_k, Y_k	Basic stochastic processes
X_k^α	Regularized exploring process
$x_k^{e,i}$	State of exploring particle i
$x_k^{a,j}$	State of auxiliary particle j
Z_k	Radon–Nykodim derivative
Z_k^a	Regularized Radon–Nykodim derivative.

II. INTRODUCTION

GPS/INS (Global Positioning System/Inertial Navigation System) integration [25] has proven to be a very efficient means of navigation due to the short term accuracy of INS allied to the long term accuracy of GPS fixes.

Tightly coupled GPS/INS units (see Fig. 1) as well as modular equipments (Fig. 2) may be found in the literature [3, 4, 19, 27] but the former is more appropriate to perform optimal signal processing. Indeed, this processing scheme allows the various errors and noise sources (such as atmospheric effects, clock delays, accelerometer biases, etc. . . .) acting on both GPS and INS units to be taken into account in a global way, as opposed to the "classical" approach of Fig. 2 where fictitious (or at least coarsely evaluated) noises are associated to the GPS fixes in the model used by the navigation filter.

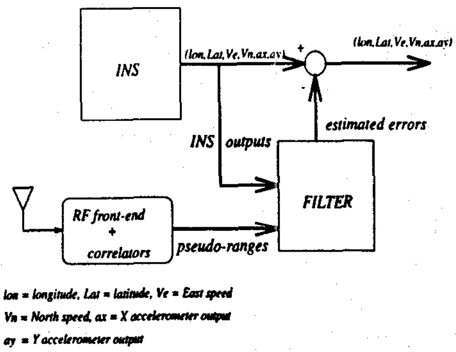

Fig. 1. Tightly coupled GPS/INS unit.

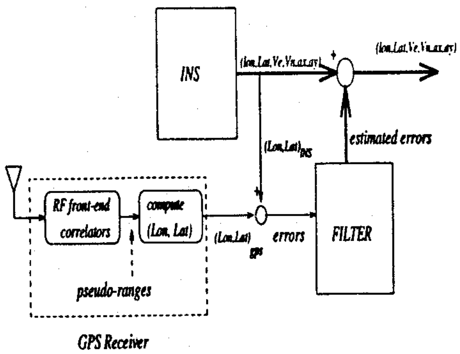

Fig. 2. Modular GPS/INS unit.

Kalman filtering [10, 11, 13, 16, 18, 21] is a popular tool in handling estimation problems, but its optimality heavily depends on linearity. When used for nonlinear filtering (extended Kalman filter (EKF)), its performance relies on, and is limited by the linearizations performed on the concerned model. On the other hand, despite early papers [2, 20, 26] on nonlinear filtering theory, the implementation of nonlinear filters has been plagued so far by the difficulties inherent to their infinite-dimensional nature.

A new approach to optimal nonlinear filtering is presented here, which is applied to the problem of INS recalibration by means of GPS. This new method, called *particle filtering* (PF), may cope with nonlinear models without any limitation, and non-Gaussian noises as well. Section III gives the mathematical basis of this new method.

The main feature of PF is that it constructs the conditional probability of the variable to be estimated, with respect to the measurements, through a suitable random particle exploration of the state space followed by a Bayes correction of the weights of the particles. Originally proposed by G. Salut, this method was first designed with birth and death of particles then with a constant number of particles, and appeared for the first time in 1989 [15]. Since then, it has been extensively studied and applied ([5, 7, 8, 22, 24]).

In this work the PF and EKF performances are evaluated by simulation of the filtering scheme of Fig. 1 in critical situations. Section IV describes the system models used by the navigations filters along with the GPS satellite constellation.

The simulation results in Section V yield a basis to compare the performance of both types of filter in terms of accumulated quadratic errors. Conclusions are drawn in Section VI.

III. DISCRETE-TIME NONLINEAR FILTERING: PARTICLE APPROACH

Nonlinear filtering [6, 16, 17, 23] concerns the recursive estimation of any function $\varphi(X_k)$ of an $\mathbb{R}^n$-valued stochastic process X from the observation of a related $\mathbb{R}^m$-valued, random process Y, where the "best" (*minimum variance*) estimator $\widehat{\varphi(X_k)}$ is given by the conditional expectation[1]

$$\mathbb{E}[\varphi(X_k) \mid Y_{\mathbf{k}} = y_{\mathbf{k}}] = \int_{\mathbb{R}^n} \varphi(x_k) P(dx_k \mid y_{\mathbf{k}})$$

where $Y_{\mathbf{k}} = (Y_0, \ldots, Y_k)$.

Let the process X be governed by

$$X_{k+1} = f(X_k, k, \omega_k) \tag{1}$$

while the observation process Y is given by

$$Y_k = h(X_k, k) + \underbrace{g(X_k, k)v_k}_{\nu_k} \tag{2}$$

where $\{\omega_k\}$ and $\{v_k\}$, $k \geq 0$, are sequences of independent random variables with appropriate dimensions, f, g, and h are measurable functions of X. We define $\nu_k = g(X_k, k)v_k$ with $R^\nu = g(X_k, k)R^v g^T(X_k, k)$, which is assumed to be strictly positive definite.

It is well known that the optimal estimator $\widehat{\varphi(X_k)}$ does not have, in general, a finitely recursive realization, as it is the case in linear-Gaussian theory (Kalman filter).

The problem of assessing $\mathbb{E}[\varphi(X_k) \mid Y_{\mathbf{k}}]$ is of course related to that of recursively computing the probability law $dP(x_{\mathbf{k}} \mid y_{\mathbf{k}})$, which provides all statistical information about X_k obtainable from the observations $Y_{\mathbf{k}} = y_{\mathbf{k}}$.

The elementary recursion from time $k-1$ to k follows two steps: prediction and correction, as explained below.

Suppose $dP(x_{\mathbf{k-1}} \mid y_{\mathbf{k-1}})$ is known. The prediction step is based on the Markovian properties of the system, and the *Chapman-Komolgorov* equation which yields

$$dP(x_{\mathbf{k}} \mid y_{\mathbf{k-1}}) = \int dP(x_k \mid x_{k-1}) dP(x_{\mathbf{k-1}} \mid y_{\mathbf{k-1}}).$$

The correction step, introduces the Bayes' ratio which may be viewed as a change of probability measure. This will be useful in the sequel. It relies on the fact [28] that there exists a random variable Z_k and a probability measure P_0 absolutely continuous with respect to P, such that X and Y become independent under P_0.

P_0 is defined by the change of measure

$$\frac{dP}{dP_0} = Z_k(X, Y)$$

and Z_k is the *Radon–Nykodim* derivative of P with respect to P_0.

Z_k may be interpreted as being the likelihood ratio relative to the hypotheses

[1] We write $dP(x_{\mathbf{k}} \mid y_{\mathbf{k-1}})$ for the correct notation $dP^{X_{\mathbf{k}}|Y_{\mathbf{k-1}}}(x_{\mathbf{k}} \mid y_{\mathbf{k-1}})$, and boldface $\mathbf{k}$ stands for the sequence $1, 2, \ldots k$.

$$\begin{cases} H_1 : Y_i = h(X_i,i) + \nu_i \\ H_0 : Y_i = \nu_i, \end{cases} \qquad \text{for} \quad i = 1,\ldots,k$$

which yields, in the case of a Gaussian observation process,

$$Z_k(X,Y) = \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{k}\|Y_i - h(X_i,i)\|_{R^\nu}^2\right\}}{\exp\left\{-\frac{1}{2}\sum_{i=1}^{k}\|Y_i\|_{R^\nu}^2\right\}} \tag{3}$$

(with notation: $\|a\|_{R^\nu}^2 = a^T[R^\nu]^{-1}a$, $a \in \mathbb{R}^m$).

The searched estimator may be expressed in terms of Z_k and P_0 as

$$\begin{aligned} \mathbb{E}[\varphi(X_k) \mid Y_{\mathbf{k}}] &= \frac{\mathbb{E}_0[\varphi(X_k)Z_k(X,Y) \mid Y_{\mathbf{k}}]}{\mathbb{E}_0[Z_k(X,Y) \mid Y_{\mathbf{k}}]} \\ &= \mathbb{E}_0[\varphi(X_k)U_k(X,Y) \mid Y_{\mathbf{k}}] \end{aligned} \tag{4}$$

where $\mathbb{E}_0$ denotes the mathematical expectation with respect to P_0, and

$$U_k = \frac{Z_k(X,Y)}{\mathbb{E}_0[Z_k(X,Y) \mid Y_{\mathbf{k}}]}.$$

It should be noted that, in this form, the filter has a *recursive* but *infinite-dimensional* solution. Further developments of filtering theory call for practical implementation methods of nonlinear filters. A rather crude approximation is the EKF.

A rather different approach to the problem is PF. The conditional expectations are estimated by means of Monte-Carlo type methods which are based on the *Law of Large Numbers*. This powerful theorem, which guarantees that the searched expectation may be approximated by the sum of independent random variables, is here specialized to the case of *conditional* probability laws.

On the basis of what was just stated, we call PF any conditional expectation approximation induced by random exploration of the state space by means of a set of independent random processes (particles) $X^{e,i}$ (where the e stands for *exploration*).

However, the conditional law $dP(x_{\mathbf{k}} \mid y_{\mathbf{k}})$ is not directly available. The introduction of the reference probability measure P_0, for which the Law of Large Numbers may be applied, yields a solution to this problem.

We introduce, as previously, a random variable Z_k defined as

$$\frac{dP}{dP_0} = Z_k(X,Y) \tag{5}$$

where dP_0 is any known and recursively realizable probability law, such as $dP(x_{\mathbf{k}})$, which yields the *a priori* sampling method, or $dP(x_{[k]} \mid x_{k-n}, y_{[k]})$ (where $x_{[k]} = [x_k, x_{k-1}, \ldots, x_{k-n+1}]$), which yields the *n-conditional* sampling method.

The basic idea is to apply the Law of Large Numbers to the numerator and denominator in (4) to obtain an estimator which converges *in probability* ($\mathbf{L}^0$) to the conditional expectation, as was demonstrated in [7].

THEOREM 1 *If the following hypothesis is valid:*
(H_1)*: Stochastic nonexplosion condition:*
There exists a nonempty class of $Y_{\mathbf{k}}$*-adapted processes* $\mathcal{X}$*, such that*

$$\begin{cases} \|\mathcal{X}_k - X_k\|_{L^2(P)} < +\infty \\ \|h(\mathcal{X}_k) - h(X_k)\|_{L^2(P)} < +\infty \end{cases}$$

and the exploring processes $X_k^{e,i}$, $(i = 1,\ldots,N)$, *belong to this class, then*

$$\frac{1}{N}\sum_{i=1}^{N} U_k^i X^{e,i} \xrightarrow[N\to\infty]{L^0(P)} \mathbb{E}_0[U_k(X,Y)X_k \mid Y_{\mathbf{k}}]$$

where $Z_k^i = Z_k(X^{e,i},Y)$ *and*

$$U_k^i = \frac{Z_k^i}{\frac{1}{N}\sum_{j=1}^{N} Z_k^j}. \tag{6}$$

We now comment on the simplest case of application.

A. A priori Exploration

If $\{X_k^{e,i}\}_{i=1,\ldots,N}$ are N random variables distributed according to $dP(x_{\mathbf{k}})$, the estimator may be written as a weighted sum of the exploring random variables:

$$\widehat{\varphi(X_k)} = \sum_{i=1}^{N} p^i \varphi(x_k^{i,e}) \tag{7}$$

where the weights $p_k^i = U_k^i$ are obtained from the Radon–Nykodim derivatives Z_k (see (3)) and, $dP(x_{\mathbf{k}})$ can be recursively obtained from $dP(x_k \mid x_{k-1})dP(x_{\mathbf{k-1}})$.

As is well known from the duality between functions and measures, this is equivalent to approximate the measure $dP(x_{\mathbf{k}} \mid y_{\mathbf{k}})$ by

$$\sum_{i=1}^{N} p^i \delta_{X_{\mathbf{k}}^{e,i}}(x_{\mathbf{k}})$$

where $\delta_{X_{\mathbf{k}}^{e,i}}(\cdot)$ denotes the Dirac's measure with pointwise support $x_{\mathbf{k}}$. That is, the positions of the particles with their weights yield a Dirac comb approximation of the searched conditional law. Fig. 3 depicts the paths of the weighted particles representing the evolution of the conditional probability law.

B. Regularization Procedure

Theorem 1 guarantees convergence for any finite k, but N may depend on k. In order to guarantee

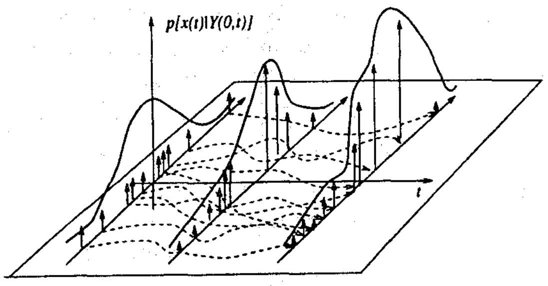

Fig. 3. A priori particles exploration of probability space.

uniform convergence with respect to (wrt) time k, we need a further regularization procedure.

To make this point clear, consider, as an example, the system

$$\begin{cases} X_k = \omega_k \\ Y_k = X_k + \nu_k \end{cases} \tag{8}$$

where ω_k and ν_k are real valued independent Gaussian random variables of unity variance, and the random variable $r_k = Z_k^{e,i}/Z_k^{e,j}$, which expresses the likelihood ratio of an exploration (sample) path i wrt another one, j.

From (2) and (3), we have

$$\mathbb{E}[r_k^q] = \left(\frac{2}{\sqrt{1-5q^2}}\right)^k \underset{k\to\infty}{\longrightarrow} +\infty.$$

That is, $\mathbb{E}[r_k^q] \to \infty$ as $k \to \infty$, since only those moments corresponding to $0 < q < 1/\sqrt{5}$ are defined. In other words, the weights p^i will numerically degenerate due to the fact that likelihoods for different sample paths diverge. We would then expect, asymptotically, null weights for almost all particles but a very few, due to normalization. Consequently, this degenerate discrete representation of the probability law bears asymptotically no useful information about the process, and some regularizing action must be carried out.

For this, consider the modified Radon–Nykodim derivative

$$\frac{dP}{dP_0^\alpha} = Z_k^\alpha(X_{\mathbf{k}}, Y_{\mathbf{k}})$$

related to the following hypothesis

$$\begin{cases} H : Y_k^\alpha = h(X_k, k) + \nu_k^\alpha \\ H_0 : Y_k^\alpha = \nu_k^\alpha \end{cases}$$

where ν_k^α is a modified Gaussian white noise with covariance $[R^{\nu,\alpha}]^{-1} = \alpha(k,l)[R^\nu]^{-1}$, $\alpha(k,l)$ is an $\mathbb{N}\times\mathbb{N} \to [0,1]$ function and ν, R^ν as defined in (2).

With the help of the above regularization procedure, a more general theorem may be stated (cf. [9]).

THEOREM 2 *Consider a regularizing function α such that*

$$\begin{cases} \alpha(0,l) = 0 \\ \alpha(k,l_0) \le \alpha(k-1,l_0), & \forall \quad l_0 \in \mathbb{N} \\ \alpha(k_0,l) \ge \alpha(k_0,l-1), & \forall \quad k_0 \in \mathbb{N} \end{cases}.$$

If the following hypotheses are valid:

(H_1^*): Uniform stochastic detectability condition: There exits a nonempty class of $Y_{\mathbf{k}}$-adapted processes $\mathcal{X}$, such that

$$\begin{cases} \|\mathcal{X}_k - X_k\|_2^* < +\infty \\ \|h(\mathcal{X}_k) - h(X_k)\|_2^* < +\infty \end{cases}$$

and the exploring processes X_k^{e,i^α}, $(i = 1,\ldots,N)$, belong to this class.

(H_2): Continuous regularizability condition

$$\hat{X}_k^\alpha \underset{\alpha\to 1}{\longrightarrow} \hat{X}_k$$

then one may choose $\alpha(N)$ such that:

$$\hat{X}_k^{\alpha(N)} = \sum_{i=1}^N \frac{Z_k^{\alpha(N)}(X^i,Y)}{\sum_{j=1}^N Z_k^{\alpha(N)}(X^j,Y)} X_k^{e,i^{\alpha(n)}}$$

$$\underset{N\to\infty}{\overset{L^0(P)^*}{\longrightarrow}} \mathbb{E}_0[U_k X_k \mid Y_{\mathbf{k}}].$$

Two examples of regularizing function are given for a Gaussian observation noise.

EXAMPLE 1 (Age-weighting factor: $\alpha(k,l) = \gamma^{k-l}$, $\gamma \in (0,1)$)

$$Z_k^\alpha(X,Y) = \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^k \gamma^{k-i}\|Y_i - h(X_i,i)\|_{R^\nu}^2\right\}}{\exp\left\{-\frac{1}{2}\sum_{i=1}^k \gamma^{k-i}\|Y_i\|_{R^\nu}^2\right\}}.$$

EXAMPLE 2 (Window of length T : $\alpha(k,l) = 1_{[0,T]}(k)$)

$$Z_k^\alpha(X,Y) = \frac{\exp\left\{-\frac{1}{2}\sum_{i=k-T}^k \|Y_i - h(X_i,i)\|_{R^\nu}^2\right\}}{\exp\left\{-\frac{1}{2}\sum_{i=k-T}^k \|Y_i\|_{R^\nu}^2\right\}}.$$

Applying the window function $\alpha(k,l) = 1_{[0,T]}(k)$ to the system (8) yields

$$\mathbb{E}[r_k^q] = \left(\frac{2}{\sqrt{1-5q^2}}\right)^T < +\infty$$

that is, the weights will not vanish, nor will the discrete representation of the conditional density degenerate.

1) *Basic A Priori Algorithm*: The *a priori* sampling algorithm may be summarized as follows.

a) Initialization. Positions of particles are initialized according to $dP(x_0)$, and weights to $1/N$.
b) Evolution. Move particles according to (1), and randomly generated noises ω_k.
c) Weighting. Weights are given by (6), and regularizations according to Examples 1 or 2.
d) Estimation. Given by (7).
e) Recursion. Step from b to d.

C. n-Conditional Exploration

While the *a priori* sampling method is well suited for stable systems, it is not the case for unstable ones, since the exploring particles tend to move away from each other, and from the real process, thus failing to satisfy hypothesis (H_1^*). The *n-conditional* method, described below, tracks the process X_k by conditioning the exploration paths to limited sections of the observed path, to avoid divergence from the real process. It should be noticed that $n = \dim(X)$ is always sufficient, if the dynamic system (1)–(2) is uniformly observable.

Consider the following partition of the time interval $[0,k]$ in $T-1$ intervals I_j, $(j = 2,\ldots,T)$ of length n and one interval I_1, of length $p < n$ such that $k = n(T-1) + p$:

$$\to \underset{0}{|} \to \underbrace{\underset{1}{]} \longrightarrow \underset{p}{]}}_{I_1}\underbrace{\longrightarrow \underset{p+n}{]}}_{I_2} \to \cdots \to \underbrace{\underset{k-n}{]} \longrightarrow \underset{k}{]}}_{I_T} \to t.$$

This is equivalent to having $I_j = \{p + (j-1)n + 1,\ldots,p + jn\}$, and we note $x_{[j]} = \{x_i\}_{i\in I_j}$, and $x_{[0]} = x_0$.

We want to sample $X_{[k]}$, conditionally to n observations $Y_{[k]}$ and X_{k-n}. The corresponding probability law appears in the following decomposition

$$\begin{aligned} dP(x_{\mathbf{k}}, y_{\mathbf{k}}) &= \prod_{j=1}^{T} dP(x_{[j]}, y_{[j]} \mid x_{[j-1]}, y_{[j-1]}) \\ &= \prod_{j=1}^{T} dP(x_{[j]} \mid x_{[j-1]}, y_{[j]}) dP(y_{[j]} \mid x_{[j-1]}) \end{aligned}$$

and we have

$$\begin{cases} dP_0 = \prod_{j=1}^{T} dP(x_{[j]} \mid x_{[j-1]}, y_{[j]}) \\ Z_k(x_{\mathbf{k}}, y_{\mathbf{k}}) = \prod_{j=1}^{T} dP(y_{[j]} \mid x_{[j-1]}). \end{cases}$$

But, since

$$\begin{aligned} &dP(x_{[k]} \mid x_{[k-1]}, y_{[k]}) \\ &\quad = \frac{dP(y_{[k]} \mid x_{[k]}, x_{[k-1]})}{dP(y_{[k]} \mid x_{[k-1]})} dP(x_{[k]} \mid x_{[k-1]}) \end{aligned}$$

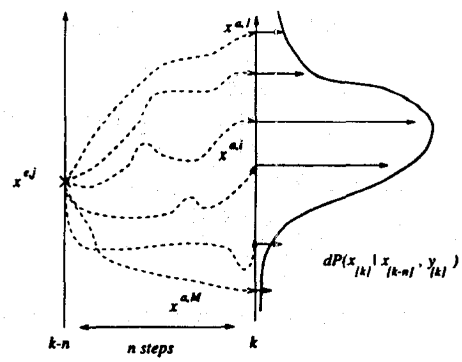

Fig. 4. Conditional exploration.

one may sample $dP(x_{[k]} \mid x_{[k-1]}, y_{[k]})$ by M particles (as in the *a priori* case) $\{x_{[k]}^{a,i}\}_{i=1,\ldots,M}$ as follows.

Given a trajectory portion $X_{[k-1]}$, the next portion is estimated, conditionally to the observations $y_{[k]}$ by auxiliary particles, moving according to $dP(x_{[k]} \mid x_{[k-1]})$ (that is, (1)), and weighted proportionally to

$$Z_{[k]}^i = \frac{dP(y_{[k]} \mid x_{[k-1]}, x_{[k]} = \{x_{[k]}^{a,i}\})}{\frac{1}{M}\sum_{l=1}^{M} dP(y_{[k]} \mid x_{[k-1]}, x_{[k]} = \{x_{[k]}^{a,l}\})}.$$

Finally, one of the M trajectory sections is randomly selected according to the newly generated probability law.

Fig. 4 illustrates this concept. It shows one of the N basic particles and its M auxiliary particles.

In terms of algorithm, this technique modifies Steps b and c of the *a priori* algorithm as shown below.

b) Evolution. For each of the N particles $\{x^{e,j}\}_{j=1,\ldots,N}$:

1) Initialize the auxiliary particle $\{x_{k-n}^{a,i}\}_{i=1,\ldots,M}$ positions and weights with those of $x_{k-n}^{e,j}$.
2) Move the auxiliary particles n steps ahead, according to (1) and its randomly generated driving noises.
3) For each of the N particles $\{x^{e,j}\}_{j=1,\ldots,N}$, weights are recursively calculated by (6).
4) Choose randomly one of the auxiliary trajectory sections using the discrete representation of the probability law expressed by the particles at instant k.

c) Weighting. The particles' weights $dP(y_{[k]} \mid x_{[k-1]})$ can be assessed by

$$dP(y_{[k]} \mid x_{[k-1]}) = \int P(y_{[k]} \mid x_{[k]}) dP(x_{[k]} \mid x_{[k-1]}),$$

(Chapman–Kolmogorov).

If we substitute $dP(x_{[k]} \mid x_{[k-1]})$ by its particle approximation $\sum_{i=1}^{N} \delta_{X_{[k]}^{e,i}}$, we have

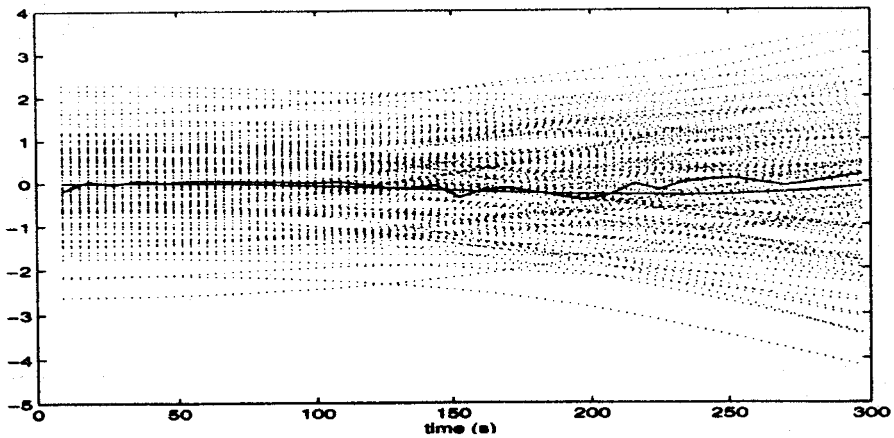

Fig. 5. A priori PF of unstable system.

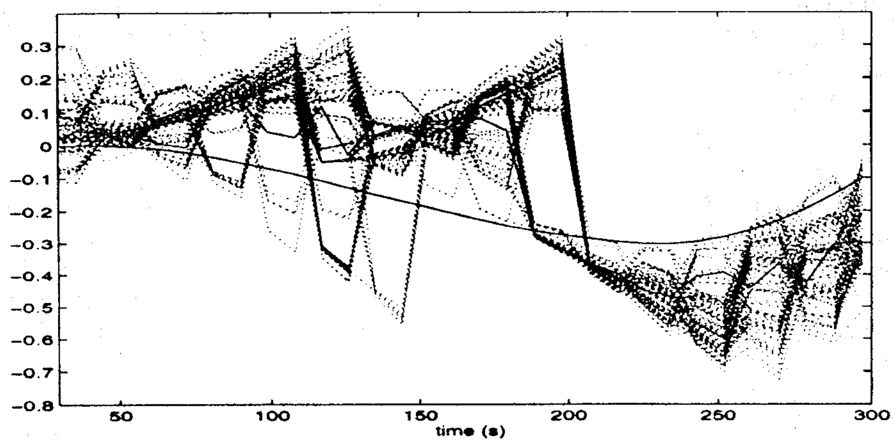

Fig. 6. A priori PF of unstable system with redistribution.

$$dP(y_{[k]} \mid x_{[k-1]}) = \sum_{i=1}^{N} \int P(y_{[k]} \mid x_{[k]}) \delta_{X_{[k]}^{e,i}}$$

$$= \sum_{i=1}^{N} P(y_{[k]} \mid X_{[k]}^{e,i})$$

and the base particle weight may be expressed in terms of auxiliary particles as follows:

$$p^{e,i} = \frac{\sum_{j=1}^{M} \{p^{a,j}\}^{i}}{\sum_{l=1}^{N} \sum_{j=1}^{M} \{p^{a,j}\}^{l}}$$

where $\{p^{a,j}\}^i$ is the weight of the auxiliary particle j corresponding to the base particle i.

D. Extensions of the Algorithm

The regularization technique introduced in the preceding paragraphs prevents the total degeneracy of the weights of the particles, but it does not prevent some of these weights from being very low relative to those of other particles, and therefore poorly contributing to the calculation of the estimator. (The application of the generalized Law of Large Numbers needs less particles as their weights are closer to $1/N$.)

In order to improve this, a redistribution technique may be applied which accelerates the (already granted) convergence. This technique is an extension of well-known sampling/resampling principles, as in [12].

It consists of periodically redistributing the particle positions (in the state space), in accordance to the discrete representation of the probability law, and reinitializing all of them with weight $1/N$. As a consequence, redistribution has a regularizing effect.

This procedure allows "heavy" particles to give birth to more particles, at the expense of light particles which die. This guarantees an occupation of the state space regions proportional to their probability mass, thus providing an adaptive grid. It has also a stabilizing effect, by eliminating particles that are too unlikely with respect to the measurements, so that it may often be used instead of n-conditional exploration. (It should be noticed, however, that uniform convergence of particle filtering has only been proved under conditions of Theorem 2.)

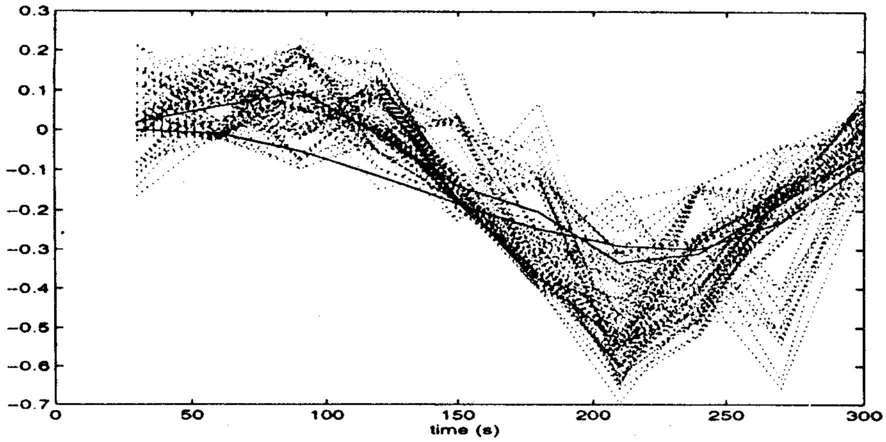

Fig. 7. N-conditional PF of unstable system with redistribution.

In order to illustrate these concepts, the following figures show an unstable system trajectory along with the exploring particles trajectories.

The first example (Fig. 5) illustrates the *a priori* method, and the divergence of particles.

Fig. 6 shows that the introduction of redistribution into the preceding case recenters the particles around the trajectory of the system, despite piecewise instability, since particles have an increasing tendency to spread away from the tracked trajectory.

Finally, Fig. 7 shows the application of the n-conditional algorithm, with redistribution. In this case, particles are kept around the trajectory of the system in a stationary fashion at all times.

IV. MODELS FOR GPS AND INS

This section describes the mathematical models used by PF and EKF as well as the simulation model used to study their performances.

A. WGS84 Reference Frame

The general reference frame in which all equations are written is the WGS84 ellipsoid defined by the following parameters:

Parameter	*Value*	*Description*
a	6.378.137, 0 m	semimajor axis
b	6.356.752, 314 m	semiminor axis
ω_E	7.292.115, $0E-11$ rad.s^{-1}	angular velocity of the Earth
μ	3.986.005, $0E8$ m^3.s^{-2}	gravitational constant

B. GPS NAVSTAR Constellation

Almanac data (see below for an excerpt of an almanac file), generally used by receivers in searching tasks, provide a good and simple approximation of the satellite orbits in the WGS-84 frame.

********* Week 733 almanac for PRN-01 *********	
ID:	01
Health:	000
Eccentricity:	3.4704208374E − 003
Time of Applicability(s):	32768.0000
Orbital Inclination(rad):	0.9549714327
Rate of Right Ascen(r/s):	−7.8746138499E − 009
SQRT(A) (m^1/2):	5153.623047
Right Ascen at TOA(rad):	1.4436991215E + 000
Argument of Perigee(rad):	−1.125461102
Mean Anom(rad):	−1.3702572584E + 000
Af0(s):	−3.8146972656E − 006
Af1(s/s):	−3.6379788071E − 012
week:	733

These data along with motion equations (9) are used to position the GPS satellites in their orbits at time t counted from the beginning of the GPS week:

$$\begin{aligned} M(t) &= M_0 + n(t - t_a) \\ \Omega(t) &= \Omega_0 + \dot{\Omega}(t - t_a) - \omega_E(t - t_{\text{week}}) \\ \delta(t) &= a_0 + a_1(t - t_a) \end{aligned} \tag{9}$$

where

$M(t), (M_0)$ is mean anomaly at time $t, (t_a)$,
$\Omega(t), (\Omega_0)$ is ascending node at time $t, (t_a)$,
n is $\sqrt{\mu/a^3}$, mean motion,
t_a is reference epoch within GPS week (seconds),
t_{week} is beginning of GPS week.

C. GPS Receiver

In a tightly coupled GPS/INS unit, the primary tasks accomplished by a GPS receiver are as follows:

—identification of all visible satellites with possible choice of the best four in terms of geometric dilution of precision (GDOP),

—measurement of satellite-to-receiver pseudo-ranges and decoding of the navigation message,

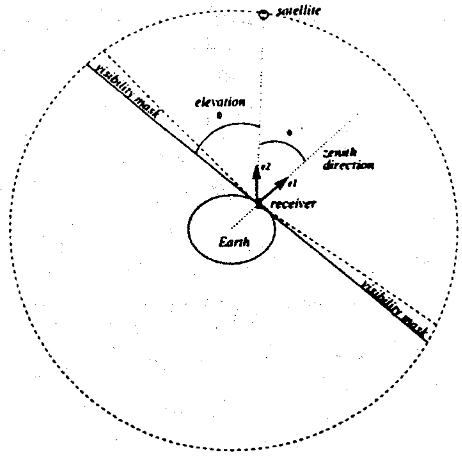

Fig. 8. Visible satellite.

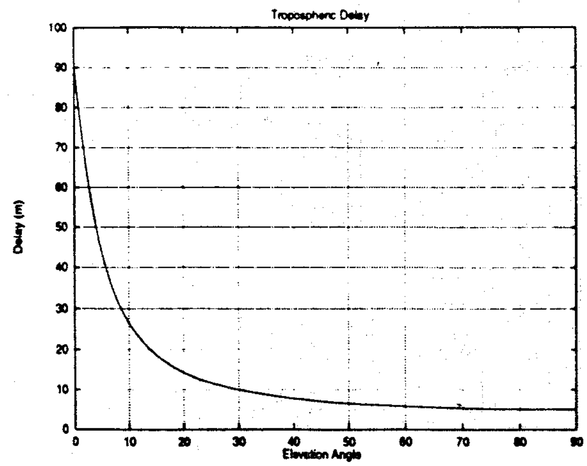

Fig. 9. Tropospheric delay.

— delivering of these data to the navigation processor.

These tasks are discussed in the following two sections.

1) *Visible Satellites*: All satellites with *elevation angle* θ greater than a user chosen *visibility mask* are considered to be visible, as shown in Fig. 8.

2) *Pseudo-Ranges*: Pseudo-ranges are modeled as the true range between satellite and receiver, corrupted by the user equipment clock bias and atmospheric propagation delay:

$$\rho_i = \sqrt{(X_{s_i} - X_m)^2 + (Y_{s_i} - Y_m)^2 + (Z_{s_i} - Z_m)^2} + c\,\Delta\tau_m + \Delta_{atm} \qquad (10)$$

where $(X_{s_i}, Y_{s_i}, Z_{s_i})$ and (X_m, Y_m, Z_m) are the S_i satellite and receiver coordinates in the WGS-84 frame, $\Delta\tau_r$ is the receiver clock bias, Δ_{atm} the atmospheric propagation delay converted in distance along the propagation path, and c is the speed of light.

By reading the navigation message, the receiver can compute the coordinates of each satellite by means of the broadcasted ephemeris data. In the simulation, these data are substituted by almanac data, and the triplets $(X_{s_i}, Y_{s_i}, Z_{s_i})$ are found by resolution of the following equations for each visible satellite:

$$E(t) = M(t) + \sin E(t) \quad \text{(Kepler's Equation)}$$

$$v(t) = 2\arctan\left[\sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\frac{E(t)}{2}\right]$$

$$\rho = A(1 - \epsilon\cos E)$$

$$X_{\text{orbit}} = \rho\cos(v + \omega)$$

$$Y_{\text{orbit}} = \rho\sin(v + \omega)$$

$$X = X_{\text{orbit}}\cos\Omega - Y_{\text{orbit}}\sin\Omega\cos i$$

$$Y = X_{\text{orbit}}\sin\Omega + Y_{\text{orbit}}\cos\Omega\cos i$$

$$Z = Y_{\text{orbit}}\sin i.$$

3) *Modeling of Drifts and Uncertainties*: The receiver clock drift δ_t is represented by the integration of an exponentially correlated random process x_t:

$$\begin{cases} \dot{x}_t = -a x_t + w_t \\ \dot{\delta}_t = x_t \end{cases} \qquad (11)$$

with $a = 1/500$ and w_t being a Gaussian white noise, $\sigma_w = 10^{-12}$ to model a typical frequency drift rate of 10^{-9} s/s for a quartz TCXO.

The atmospheric delays are caused mainly by the propagation of the electromagnetic wave through the ionospheric and tropospheric layers. Since the former effect can be overcome by double-frequency receivers, the atmospheric delays were represented by a simplified version of Goad and Goodman tropospheric model [14]. The tropospheric delay $\Delta_{\text{tropo}}(\theta)$ is shown in Fig. 9 as a function of the elevation angle θ. The value of the delay, converted into meters, is given by

$$\Delta_{\text{tropo}}(\theta) = 10^{-6}\{N_{d,0}^{\text{tropo}}(\sqrt{(R+h_d)^2 - R^2\cos^2\theta} - R\sin\theta) + N_{w,0}^{\text{tropo}}(\sqrt{(R+h_w)^2 - R^2\cos^2\theta} - R\sin\theta)\}$$

where R is the Earth's radius, h_d and h_w are the heights of the dry and wet tropospheric layers:

$$h_w \approx 11 \text{ km}$$

$$h_d = 40136 + 148.72(T - 273.16) \text{ [m]}$$

with refraction indices:

$$N_{d,0}^{\text{tropo}} = 77.64\frac{p}{T}$$

$$N_{w,0}^{\text{tropo}} = -12.96\frac{e}{T} + 3.71810^5\frac{e}{T^2}$$

where

p is local atm. pressure (1013.0 mbar),
T is local temperature (298.0 K),
e is partial pressure of water vapor (34.0 mbar).

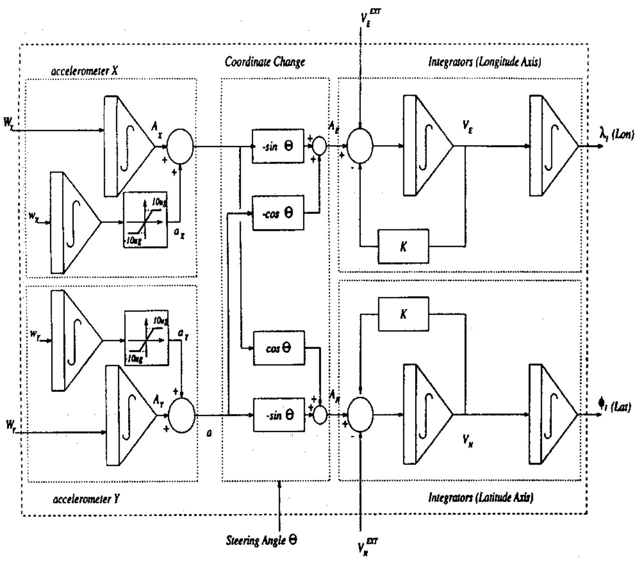

Fig. 10. INS axis model.

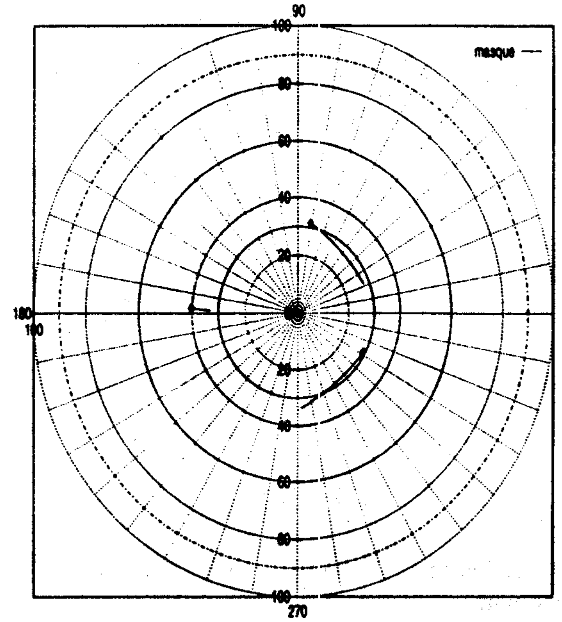

Fig. 11. Skyplot.

A Gaussian white noise with a standard deviation equal to 5% of $\Delta_{\text{tropo}}(\theta)$ is added to the pseudo-ranges to take uncertainties of the model into account.

These data are delivered to the navigation processor, which calculates the navigation solution, in view of the data gathered from INS outputs.

D. INS Model

The INS [1, 11] is represented as a two-axis[2] strap-down unit, shown in Fig. 10. The X and Y accelerometers are assumed to lie in the local tangent plane to the Earth, and the mobile's steering direction (Θ) of the mobile wrt North is assumed to be perfectly determined. This is equivalent to having the accelerometers as the sole source of error.

In this simplified model, the accelerometers deliver biased acceleration information along the X and Y axes ($A_X + a_X$ and $A_Y + a_Y$) where the biases are modeled as random walks, limited to $10\mu g$, so as to be representative of typical accelerometers biases. The acceleration is integrated twice to yield longitude (λ_I) and latitude (ϕ_I). A velocity feedback loop of gain K, introduces a damping factor that reduces the accumulated (integrated) effect of the bias over the position results. Exact external velocity references in the North (V_N^{ext}) and East (V_E^{ext}) directions are provided to account for this damping.

1) *INS Error Model*: If we define the vector $\mathbf{e} = [e_\lambda, e_\phi, \dot{e}_\lambda, \dot{e}_\phi, a_x, a_y]^T$ containing the INS errors in longitude (e_λ), latitude (e_ϕ), their derivatives ($\dot{e}_\lambda, \dot{e}_\phi$), and the accelerometers biases (a_x, a_y), then the *error model* for the INS may be written as

$$d\begin{bmatrix} e_\lambda \\ e_\phi \\ \dot{e}_\lambda \\ \dot{e}_\phi \\ a_x \\ a_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -K & 0 & -\cos\Theta & -\sin\Theta \\ 0 & 0 & 0 & -K & -\sin\Theta & \cos\Theta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_\lambda \\ e_\phi \\ \dot{e}_\lambda \\ \dot{e}_\phi \\ a_x \\ a_y \end{bmatrix} dt + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d\omega_x \\ d\omega_y \end{bmatrix} \tag{12}$$

where $d\omega_x, d\omega_y$ are Brownian increments of adequate variance.

E. INS/GPS Integrated Model

In order to be useful for filtering purposes, the GPS/INS state-space model should comprise the following:

— a dynamic equation containing the laws of evolution for the system states (INS errors and receiver clock bias), such as (12) and (11),

— observation equations, where the observables (pseudo-ranges) are expressed as functions of the above mentioned states.

[2] The X axis is aligned with the steering direction of the mobile, axis Z points upward in the local vertical direction, and the Y axis completes the orthogonal system.

The evolution equations for the state vector containing the INS errors, and its derivatives, as well as the clock drift are given in discrete-time form in (13) below:

$$
\begin{bmatrix} e_\lambda \\ e_\phi \\ \dot e_\lambda \\ \dot e_\phi \\ a_x \\ a_y \\ \delta_t \\ \dot\delta_t \end{bmatrix}_{k+1}
=
\begin{bmatrix}
1 & 0 & \Delta t & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \Delta t & 0 & 0 & 0 & 0 \\
0 & 0 & 1-K\Delta t & 0 & -\Delta t\cos\Theta_{k\Delta t} & -\Delta t\sin\Theta_{k\Delta t} & 0 & 0 \\
0 & 0 & 0 & 1-K\Delta t & -\Delta t\sin\Theta_{k\Delta t} & \Delta t\cos\Theta_{k\Delta t} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \Delta t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-a\Delta t
\end{bmatrix}
\begin{bmatrix} e_\lambda \\ e_\phi \\ \dot e_\lambda \\ \dot e_\phi \\ a_x \\ a_y \\ \delta \\ \dot\delta \end{bmatrix}_k
+
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\sigma_x\sqrt{\Delta t} & 0 & 0 \\
0 & \sigma_y\sqrt{\Delta t} & 0 \\
0 & 0 & 0 \\
0 & 0 & \sigma_\delta\sqrt{\Delta t}
\end{bmatrix}
\begin{bmatrix} \Delta\omega_x \\ \Delta\omega_y \\ \Delta\omega_\delta \end{bmatrix}
\qquad (13)
$$

where the $\Delta\omega$s are unitary Brownian drift increments.

If (λ_m, ϕ_m) are the coordinates of the mobile, and (λ_I, ϕ_I) are the corresponding INS outputs, we may write

$$
\begin{cases} \lambda_I = \lambda_m + e_\lambda \Rightarrow \hat\lambda_m = \lambda_I - \hat e_\lambda \\ \phi_I = \phi_m + e_\phi \Rightarrow \hat\phi_m = \phi_I - \hat e_\phi \end{cases}. \qquad (14)
$$

The observation equation for a visible satellite S_i may be then written as

$$
\begin{aligned}
\rho_i = {} & h(X_{S_i}, Y_{S_i}, Z_{S_i}, \lambda_I, \phi_I, e_\lambda, e_\phi) - c\delta \\
& + g(X_{S_i}, Y_{S_i}, Z_{S_i}, \lambda_I, \phi_I, e_\lambda, e_\phi)[1 + \sigma_{\text{tropo}}\nu]
\end{aligned} \qquad (15)
$$

where

$$
\begin{aligned}
& h(X_{S_i}, Y_{S_i}, Z_{S_i}, \lambda_I, \phi_I, e_\lambda, e_\phi) \\
& \quad = \sqrt{(X_{S_i} - X_m)^2 + (Y_{S_i} - Y_m)^2 + (Z_{S_i} - Z_m)^2} \\
& g(X_{S_i}, Y_{S_i}, Z_{S_i}, \lambda_I, \phi_I, e_\lambda, e_\phi) \\
& \quad = \Delta_{\text{tropo}}(\theta_i)
\end{aligned}
$$

$\sigma_{\text{tropo}}\nu$ = white noise with std. dev. equal to 5% of $\Delta_{\text{tropo}}(\theta_i)$.

The observation equation is clearly nonlinear due to square root terms containing coordinate changes in the WGS-84 frame:

$$
\begin{cases}
X_m = (R_p + h)\cos(\phi_I - e_\phi)\cos(\lambda_I - e_\lambda) \\
Y_m = (R_p + h)\cos(\phi_I - e_\phi)\sin(\lambda_I - e_\lambda) \\
Z_m = \left(\dfrac{b^2}{a^2} R_p + h\right)\sin(\phi_I - e_\phi)
\end{cases}
$$

where

$$
R_p = \frac{a^2}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}
$$

is the radius of curvature in the prime vertical, and the tropospheric delay dependency on the elevation angle:

$$
\theta_i = \frac{\pi}{2} - \arccos \times \left[\frac{Xm(X_{S_i} - X_m) + Ym(Y_{S_i} - Y_m) + Zm(Z_{S_i} - Z_m)}{\sqrt{Xm^2 + Ym^2 + Zm^2}\sqrt{(X_{S_i} - X_m)^2 + (Y_{S_i} - Y_m)^2 + (Z_{S_i} - Z_m)^2}}\right].
$$

The linearization used in the EKF, for comparison purposes, follows the one presented in [29], where λ and ϕ are substituted as in (14) and the differentiations are made with relation to e_λ and e_ϕ.

V. SIMULATION RESULTS

The validation of the filtering schemes (PF and EKF) just described was checked by simulation, which included the vehicle movement, the INS outputs, the GPS readings (pseudo-ranges) and the constellation of the satellites. The models described in the later sections were used to this end.

In order to show the superiority of the PF approach, a critical situation was chosen in which the number of visible satellites decays from 3 (the minimum required for 2D static positioning) to 2. In this situation GDOP is undefined (the system of nonlinear equations is not invertible). We say that static observability no longer holds. Positioning information must be *dynamically* extracted from the pseudo-range sequence. Indeed, the theoretical observability is ensured, since the successive positions of 2 moving satellites may be thought of

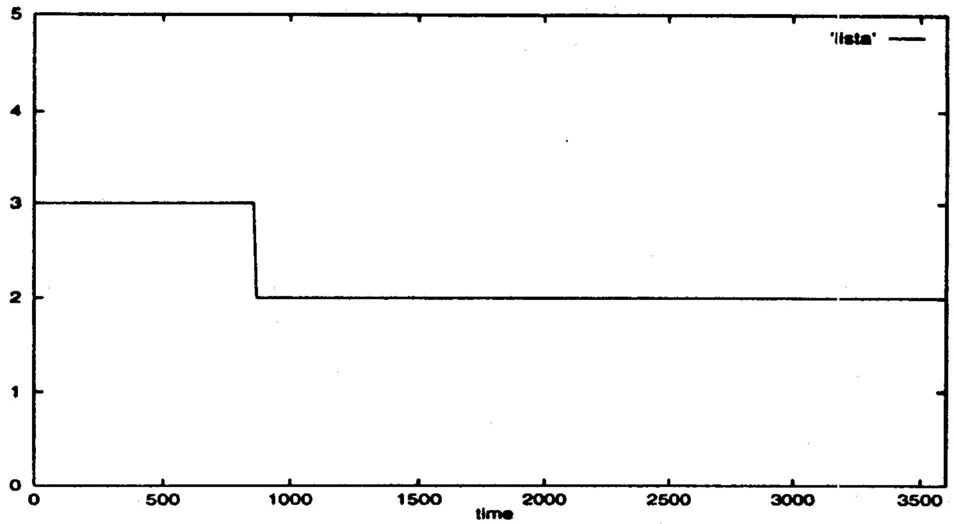

Fig. 12. Number of visible satellites.

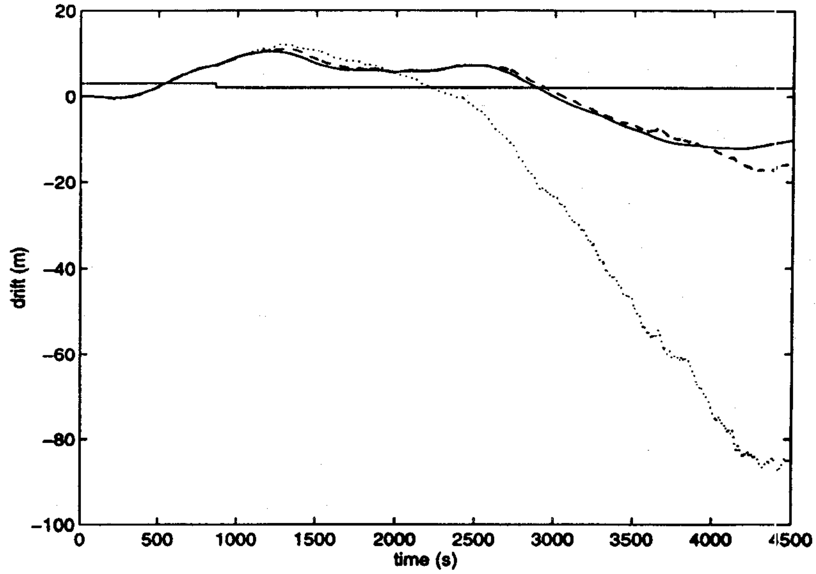

Fig. 13. Longitude drift.

as being many satellites, thus providing the necessary positioning information.

Pseudo-ranges were available at an interval of 1 s, and total simulation duration was 3600 s.

An *a priori* PF[3] with $N = 1000$ particles was used in the simulation. The period of redistribution varies automatically. (Redistribution occurs when the number of particles whose weights are smaller than $1/(10N)$ is higher than 10% of the total number of particles N.)

Fig. 11 shows the *sky-plot* corresponding to the simulation period. A visibility mask of 50° was chosen to provide the desired critical situation. The number of visible satellites is shown in Fig. 12.

The following Figs. 13–18 show the filters outputs, i.e., the estimations of the INS drifts (solid line) in longitude, latitude and time, along with the corresponding rms errors, as estimated by the PF (dashed line) and EKF (dotted line). The rms error for a variable X is given by

$$e_t^X = \sqrt{\frac{1}{t}\sum_{\tau=0}^{t}(X_\tau - \hat{X}_\tau)^2}.$$

It is clear from these results that, as the satellites disappear below the visibility mask, the EKF can no longer keep track of the INS drifts, since it cannot

[3]Although not formally demonstrated, experience shows the equivalence, concerning uniform convergence, of the *a priori* filter with redistribution and the n-conditional filter, *under large observation noise covariance conditions*.

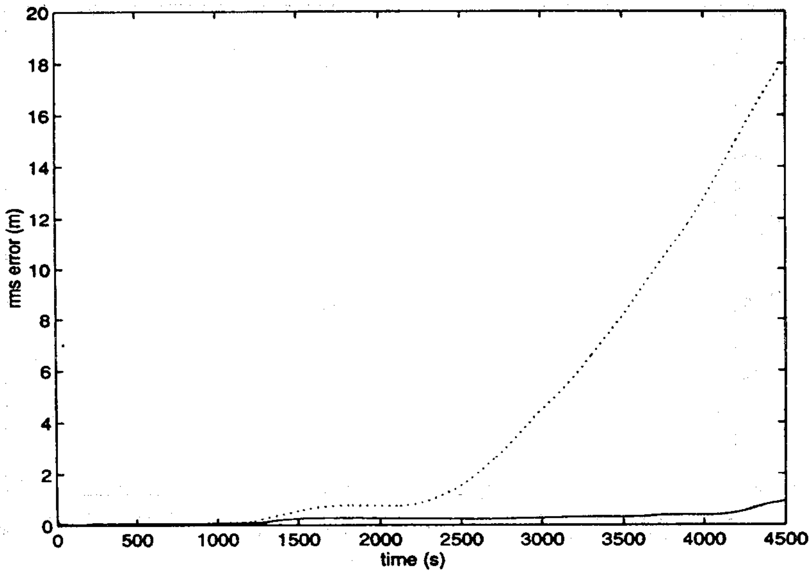

Fig. 14. RMS estimation error on longitude drift.

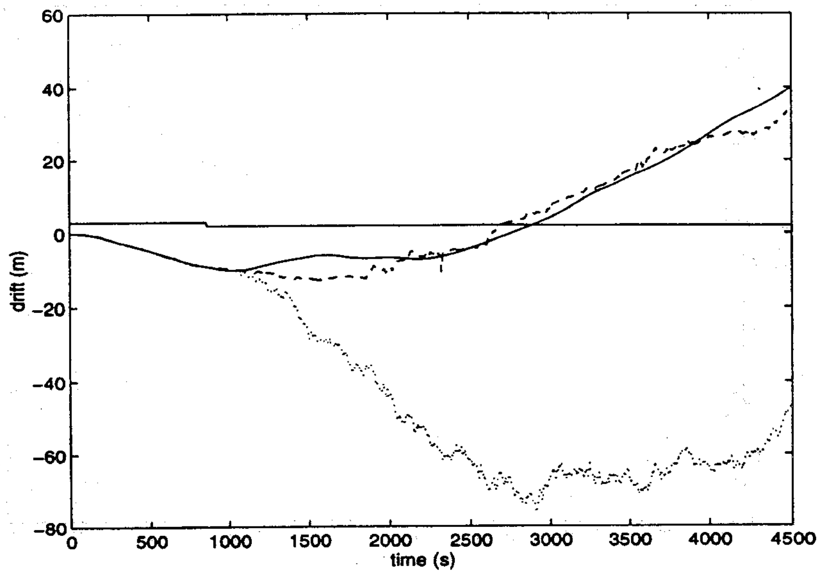

Fig. 15. Latitude drift.

correctly exploit the dynamic observability of the system, deteriorated by linearization.

In normal situations (sufficient number of visible satellites), where static observability is ensured, the EKF linearization, which consists of replacing the satellites centered spheres by tangent planes, may be considered a good approximation of the problem, and both filters are expected to exhibit similar performances.

VI. CONCLUSIONS

These results show the clear superiority of *particle* nonlinear filtering over extended Kalman filtering in critical situations, although the former is more time/memory consuming, as the number of particles grows. These problems are overcome by the advent of new technologies, making parallel processing available to embedded systems, and enabling PF to be implemented in on-board real-time systems.

The efficiency of the PF approach stems from the uniform convergence of the particle approach as $N \rightarrow \infty$. The degradation caused by finite number of particles and unstable models is compensated by redistribution and conditional evolution techniques, respectively.

The PF method can be seen from now on as a general purpose, nonlinear filtering method, which

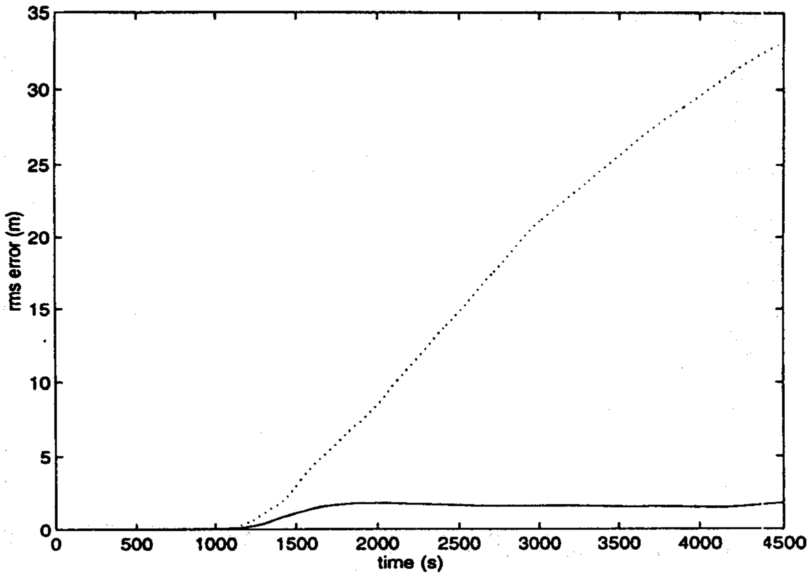

Fig. 16. RMS estimation error on latitude drift.

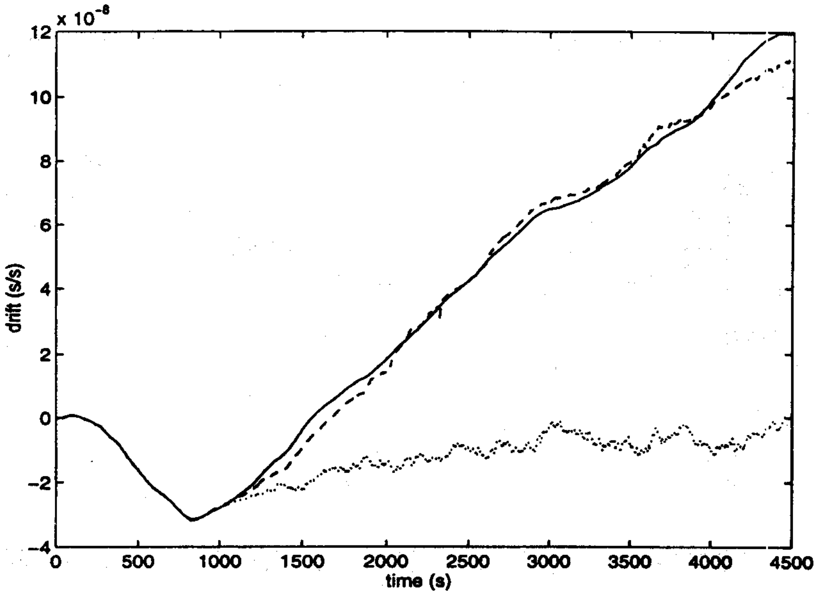

Fig. 17. Clock drift.

discards linearizations or series approximation. These techniques modify the system model itself in order to solve the filtering problem. The PF technique instead, approaches the solution using the best available model, and encourages realistic nonlinear modeling instead of simplifications.

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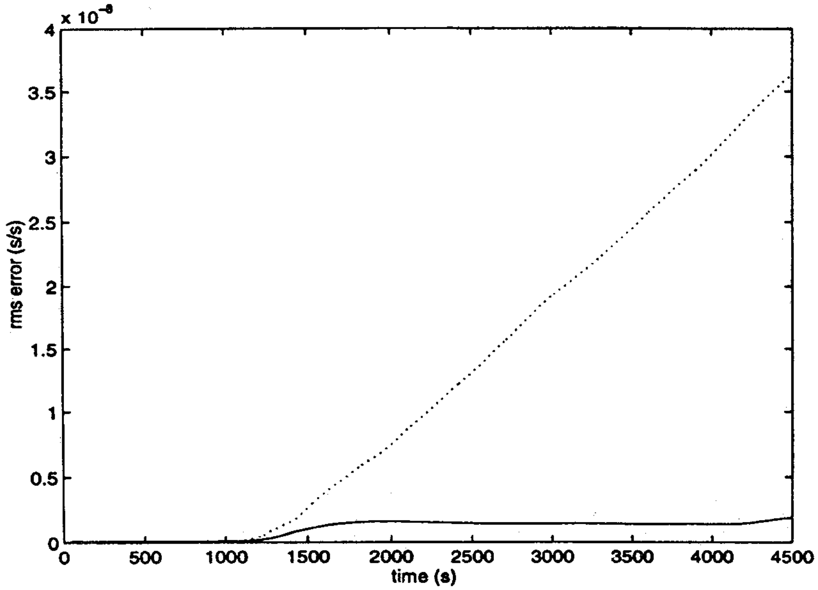

Fig. 18. RMS estimation error on clock drift.

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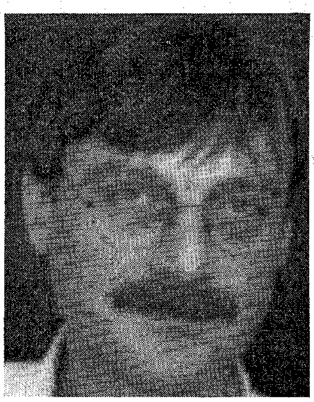
Himilcon Carvalho was born in Vitória, Brazil, and received his B.S. degree in electrical engineering from Instituto Tecnológico de Aeronáutica, S. José dos Campos, Brazil, his DEA and D.Sc. degrees from ENSAE, Toulouse, France, in 1986, 1993, and 1995, respectively.

From 1986 to 1991 he worked as a Spacecraft Systems Engineer in the development of the Brazilian Space Program (MECB) at the National Institute for Space Research, INPE. After a specialization period on Space Telecommunications at ENSAE in 1992 (Toulouse, France) he worked with the Optimization, Filtering and Parallel Processing group at Laboratoire d'Analyse et d'Architecture des Systèmes, LAAS/CNRS, Toulouse, as well as DIGILOG, Aix-en-Provence, France.

His current research interests are in the areas of optimal nonlinear filtering, inertial navigation, signal processing, and GPS.

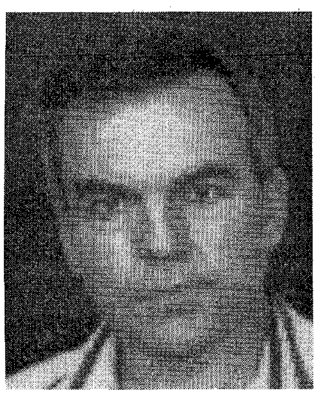
Pierre del Moral graduated from University Paul Sabatier (U.P.S.) of Toulouse in pure mathematics and received the D.E.A. degree in 1989. He received the D.Sc. degree in automation and signal processing from U.P.S. in 1994.

From 1990 to 1995, he was a researcher with the LAAS-CNRS as well as DIGILOG. In October 1995, he joined the Probability and Statistics Department of the University Paul Sabatier, where he is currently a CNRS-researcher. His research activities have been in the general areas of signal processing, stochastic processes, optimal control, and idempotent analysis.



André Monin was born in Le Creusot, France in 1958. He graduated from the Ecole Nationale Supérieure d'Ingénieurs Electriciens de Grenoble in 1980.

From 1981 to 1983, he was a teaching assistant in the Ecole Normale Supérieure de Marrakech, Morocco. Since 1985, he has been with the LAAS-CNRS, France, as "Chargé de Recherche" from 1989. After some work on nonlinear systems representation for his Doctoral thesis at Université Paul Sabatier, Toulouse, France, in 1987, his interests moved to the area of nonlinear filtering, systems realization and identification.



Gérard Salut was born in 1947 in Toulouse, France. He graduated as an engineer from the Institut des Sciences Appliquées, Toulouse, in 1968. He obtained the degree of Docteur d'Etat from the University Paul Sabatier of Toulouse in 1976, and the "Médaille de Bronze" of CNRS in 1977.

After a fellowship at the Institute of Computer Science of the University of London, he was "Chargé de Mission" for the Scientific Affairs Department of the French Embassy in London, up to 1972. From 1977 to 1978, he was NSF-CNRS Exchange Fellow at the Electronics Research Laboratory of the University of California, Berkeley. He is now Director of Research at the LAAS-CNRS, in Toulouse, France. His areas of research include nonlinear signal processing, identification and model reduction, as well as particle resolution of filtering and optimization problems.