Particle Methods for Filtering & Uncertainty Propagations

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Some references :

• Feynman-Kac formulae. Genealogical and interacting particle approximations. Springer New York, Series: Probability and Applications (04).
• Particle Methods: An introduction with applications. HAL-INRIA-699 (09).
• Backward Particle Interpretation of Feynman-Kac models HAL-INRIA-7019 (09).
• Concentration Inequalities for Mean Field Particle Models HAL-INRIA-6901(09).

The filtering problem $\subset$ Bayesian statistics

- X_t**:=Signal=Stochastic process**

  Engineering/physics/biology/economics :

  - Non cooperative targets (defense : missile, boat, plane,...).
  - Physics (Fluids : twisters, cyclones, ocean models, pressure/temperature/diffusion coefficients,...).
  - Finance (assets, portfolios, volatilities, default indexes,...).
  - Signal (speech, codes, informations transmissions, waves,...).

  Dynamics and sources of randomness :

  - Physical evolution equations (example : $\sum_i u_i \overrightarrow{F}_i = \overrightarrow{A}$)
  - Perturbations and random sources:
    - Model uncertainties $\oplus$ External perturbations.
    - **Unknown controls and related model parameters.**

$\rightsquigarrow$ **Prior Knowledge/Law** (unknown quantities=random samples.)

The filtering model

- Y_t=**Partial and Noisy observations of the signal** X_t **:**

  Engineering/physics/biology/economics :

  - Engineering : Radar, Sonar, GPS, ...
  - Physics (sensors : pressure/temperature/...).
  - Finance (assets, portfolios,...).
  - Statistics (real data: medecine, pharmacology, politics, economics,...).

  Dynamics and sources of randomness :

  - Partial observations : complex mixtures, partial coordinates.
  - Perturbations et random sources :
    - Noisy sensor measures (thermal noise).
    - External/environmental perturbations, model uncertainties.

Objectives

Compute/Sample/Estimate inductively the flow of measures

$$\mathrm{Law}((X_0,\ldots,X_t) \mid (Y_0,\ldots,Y_t)) \quad \text{or only} \quad \mathrm{Law}(X_t \mid (Y_0,\ldots,Y_t))$$

$\oplus$ when $(X,Y) \sim \Theta$ =Unknown parameter

$$\mathrm{Law}(\Theta \mid (Y_0,\ldots,Y_t))$$

Equivalent terminologies :

- Data Assimilation (forecasting, fluids/ocean models).
- Hidden Markov Chains Models (HMM).
- Posterior Law=Law($X|Y$) (A Priori=$\mathrm{Law}(X)$).

Particle filters
= Genetic type population of N individuals=particles=samples

Current population :

$$(\xi_t^1, \ldots, \xi_t^N) \in E^N \rightsquigarrow \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_t^i} = \mathrm{Law}(X_t \mid (Y_0, \ldots, Y_t))$$

Genealogical tree based learning algorithm :

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{\textbf{i-th ancestral line(t)}} = \mathrm{Law}((X_0, \ldots, X_t) \mid (Y_0, \ldots, Y_t))$$

The adaptation/learning/filtering scheme :

- Prediction/Exploration $\rightsquigarrow$ sampling N local transitions of the signal.
- Updating/Correction/Selection $\rightsquigarrow$ branching process (fixed size N).
  - Kill/stop individuals/proposal with poor likelihood value.
  - Multiply/increase individuals with high likelihood value.

Particle filters=Mean field particle technique

- Filtering equation =Nonlinear "discrete time" PDE :

$$\mathrm{Law}(X_t \mid (Y_0, \ldots, Y_t)) := \eta_t = \Phi_t(\eta_{t-1})$$

- Mean field particle model :

If

$$\eta^N_{t-1} = \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i_{t-1}} \simeq_{N\uparrow\infty} \eta_{t-1}$$

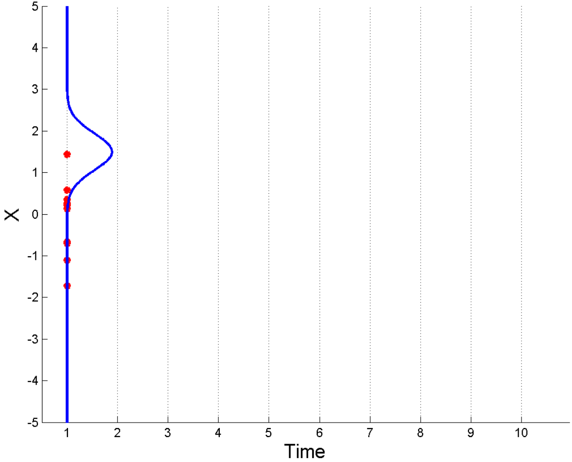
Then

$$\text{Sampling } \left(\xi^1_t, \ldots, \xi^N_t\right) \text{ i.i.d. } \Phi_t\left(\eta^N_{t-1}\right) \simeq_{N\uparrow\infty} = \Phi_t\left(\eta_{t-1}\right) = \eta_t$$

$$\Downarrow$$

$$\eta^N_t = \frac{1}{N}\sum_{i=1}^N \delta_{\xi^i_t} \simeq_{N\uparrow\infty} \eta_t$$

Illustration: Genetic genealogical tree particle models



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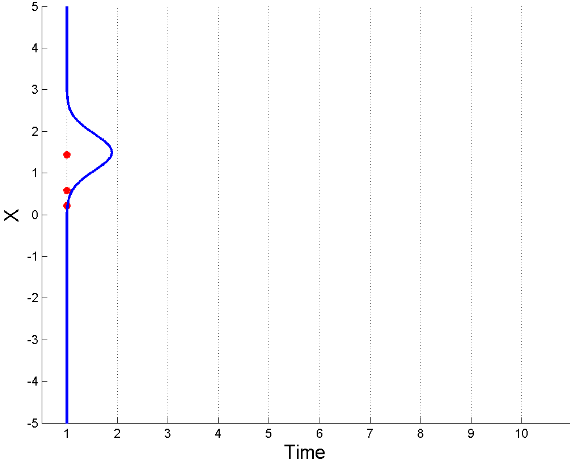

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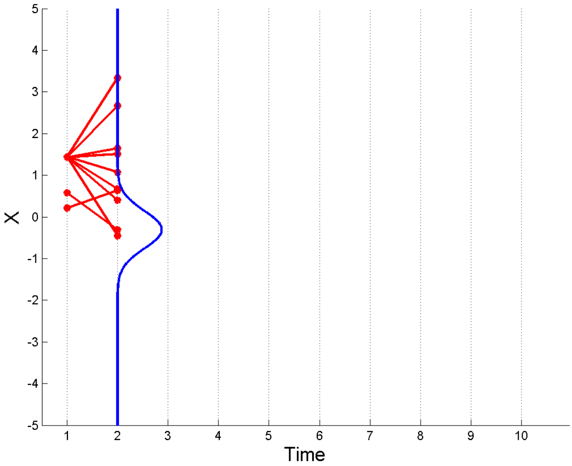

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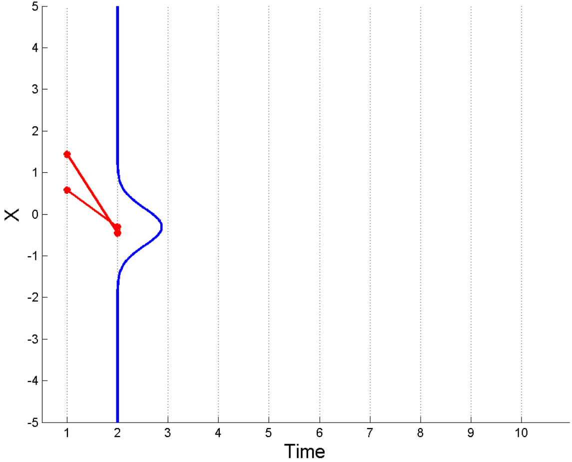

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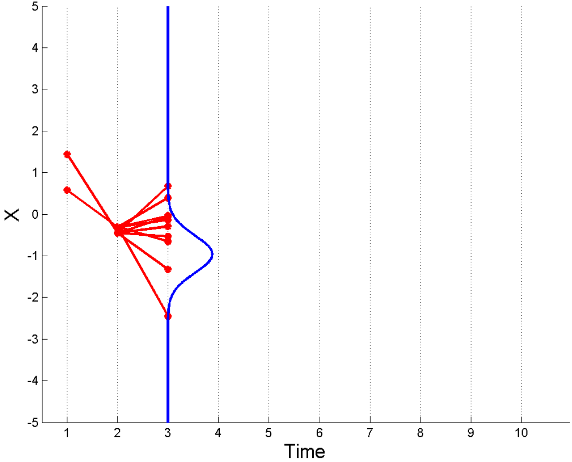

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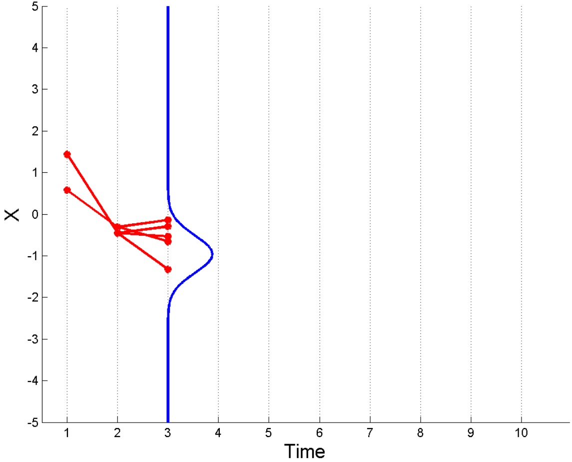

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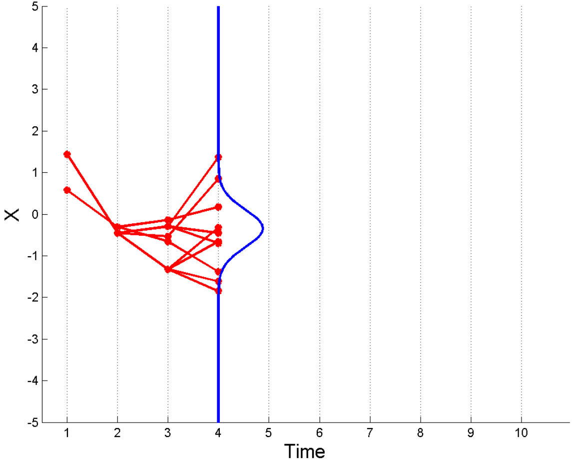

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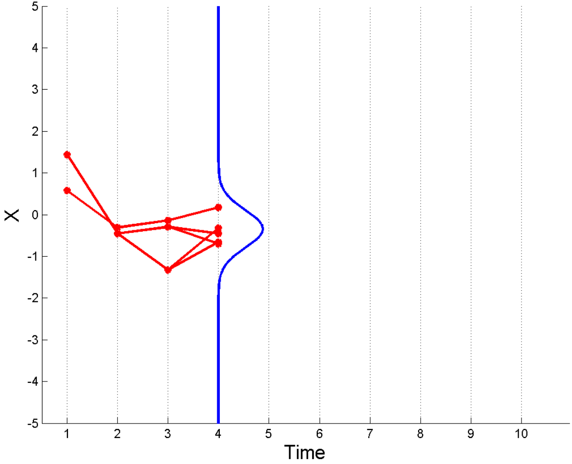

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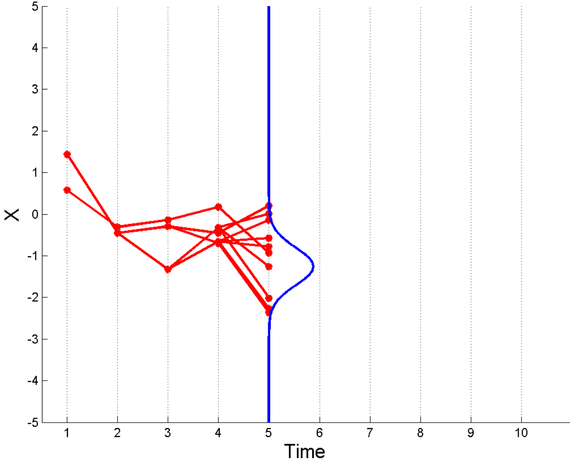

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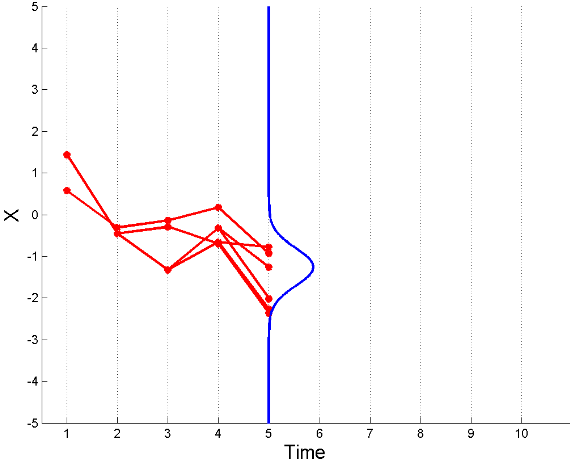

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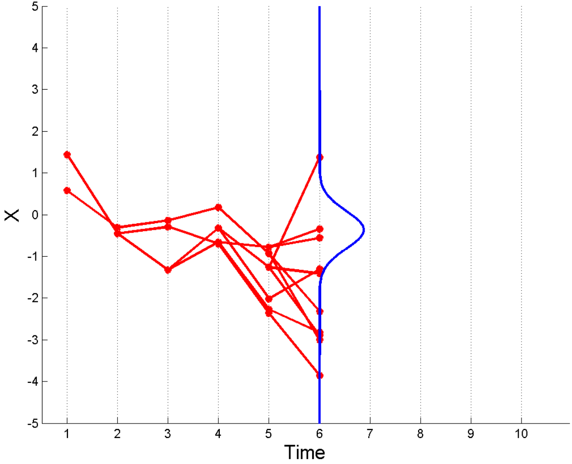

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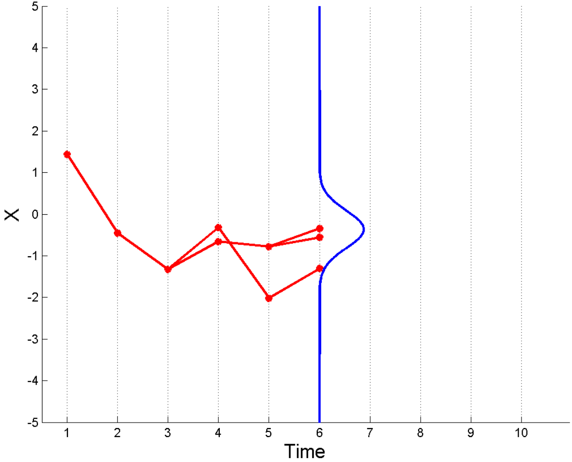

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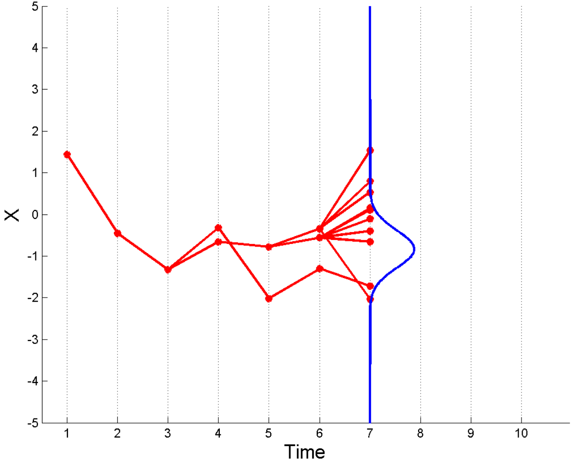

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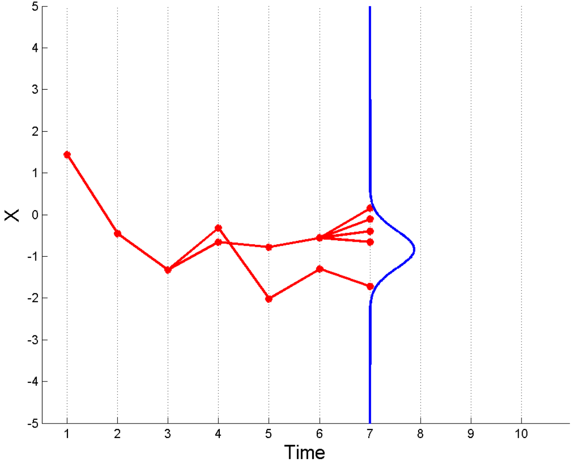

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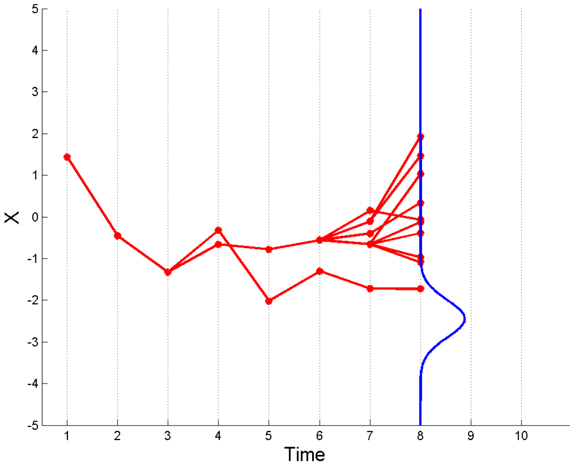

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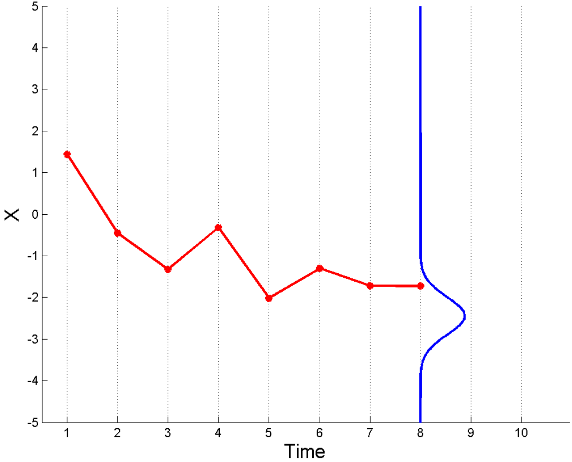

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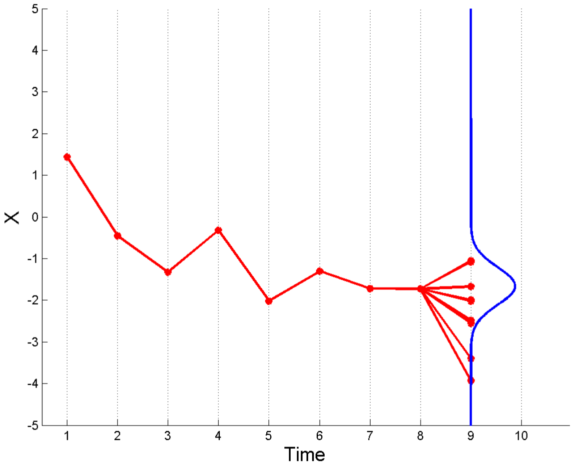

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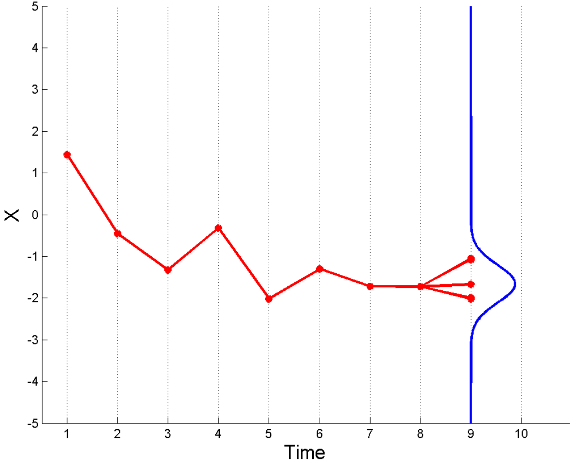

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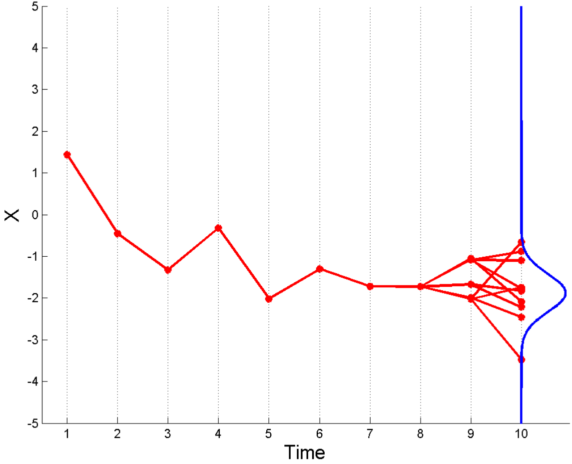

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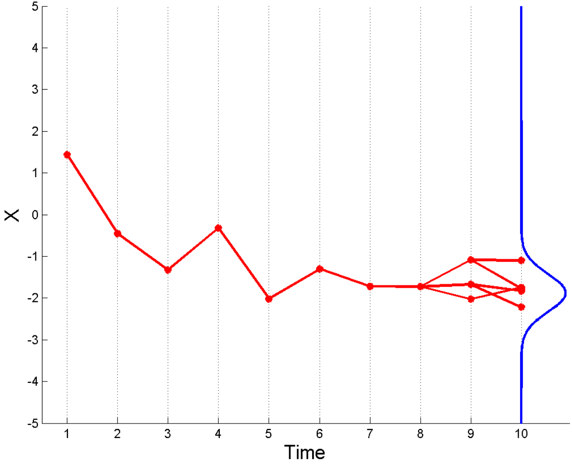

Rare event analysis

- **Stochastic process** X $\oplus$ **Rare event** A **:**

$$\mathrm{Proba}(X \in A) \quad \& \quad \mathrm{Law}((X_0,\ldots,X_t) \mid X \in A)$$

$\supset$ engineering/physics/biology/economics/finance :

- *Finance:* ruin/default probab., financial crashes, eco. crisis,...
- *engineering :* networks overload, breakdowns, failures,...
- *Physics :* climate models, directed polymer conformations, particle in absorbing medium, Schroedinger ground states,...
- *Statistics :* tail probabilities, extreme random values.
- *Combinatorics :* Complex enumeration problems.

- **Process strategies** $\in$ **Rare event** $\Rightarrow$ Control and prediction.

$$X_t = F_t(X_{t-1}, W_t) \to \mathrm{Law}((W_0,\ldots,W_t) \mid X \in A)$$

Flow of measure with increasing sampling complexity

- Rare event $=$ cascade/series of intermediate less-rare events ($\uparrow$ energy levels, physical gateways, index level crossings).
- Conditional probab.$=$ Nonlinear flow of optimal twisted measures

$$\eta_t = \text{Law}(\text{process} \mid \text{series of } t \text{ intermediate events})$$

- Rare event probabilities $=$ Normalizing constants.

Particle methods

(Sampling a genealogical type default tree model $\oplus$ % success or default)

- Explorations/Local search propositions of the solution space.
- Branching-Selection individuals $\in \uparrow$ critical regimes.

Stochastic linearization/perturbation technique

$$\eta_n^N = \Phi_n(\eta_{n-1}^N) + \frac{1}{\sqrt{N}}\; W_n^N$$

with $W_n^N \simeq W_n$ **independent and centered Gaussian fields**.

Key advantages:

- $\eta_n = \Phi_n(\eta_{n-1})$ **stable dynamical/PDE equation**

$$\Longrightarrow \quad \text{local errors do not propagate}$$

$$\Longrightarrow \quad \text{uniform control of errors w.r.t. the time parameter}$$

- "No burning, no need to study the stability of MCMC models".
- Stochastic adaptive grid approximation
- Nonlinear system $\rightsquigarrow$ "positive-benefic interactions.
- Simple and natural sampling algorithm,...

Equivalent Stochastic Algorithms :

- Genetic and evolutionary type algorithms.
- Spatial branching models, subset splitting techniques.
- Sequential Monte Carlo methods, population Monte Carlo models.
- Diffusion Monte Carlo (DMC), Quantum Monte Carlo (QMC), ...
- Some botanical names $\sim \neq$ application domain areas :

  bootsrapping, selection, pruning-enrichment, reconfiguration, cloning, go with the winner, spawning, condensation, grouping, rejuvenations, harmony searches, biomimetics, splitting, ...

$$\Updownarrow$$

1950 $\leq$ [(Meta)Heuristics] $\leq$ 1996 $\leq$ Feynman-Kac particle model