General multi-object filtering and association measure

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***Abstract*—This short paper focuses on the structure of the data association problem and details a solution based on the introduction of distinguishability in the representation of a given stochastic population. This approach allows for the derivation of general filtering equations for independent stochastic populations. Based on these general equations, the concept of association measure is defined recursively.**

INTRODUCTION

Multi-object estimation has been an active area of research for decades; the underlying challenge is to estimate the state of a varying number of objects subject to a random, and possibly spurious observation [1]. One of the most popular approaches for tackling this problem is known as the Probability Hypothesis Density filter, or PHD filter [2], [3]. It is based on Random Finite Set (RFS) theory [4] which has been successfully applied on a broad range of applications. Recent research includes the connection between spatial branching processes and the PHD recursion [5], the study of spatial cluster modelling [6], and the derivation of spatial second moment for the multi-object posterior [7], [8].

The PHD filter relies on a Poisson i.i.d. approximation for describing stochastic populations, and propagates the first moment density, or intensity, of the associated multi-object posterior. In the PHD filter, as well as in most of the filtering equations based on the concept of point processes, points are assumed to be indistinguishable, and no information about specific individuals in the population can be learned directly.

In this paper, after introducing some notations in Section I, a new approach based on the concept of distinguishability is considered for the representation of stochastic populations in Section II. Based on this concept, general multi-object filtering equations for independent populations are derived in Section III. These filtering equations are written in terms of association measure, in order to exhibit their structure.

I. NOTATIONS

Throughout the paper, the main results will be stated using the formalism of probability theory. Even though this choice is not the most usual in the area of multi-target tracking [1], the concepts and notations of probability theory are well suited to the topic of the paper and make the statement of the results simpler.

Henceforth, $\mathcal{M}(E)$ will stand for the set of finite positive measures on a given measurable space $(E, \mathcal{E})$, while $\mathcal{P}(E)$ will be the set of probability measures on $(E, \mathcal{E})$. Additionally, the Banach space of all bounded and measurable functions equipped with the uniform norm $\|\cdot\|$ will be denoted $\mathcal{B}(E)$, and we write $\mu(f) = \int \mu(\mathrm{d}x) f(x)$ for any $\mu \in \mathcal{M}(E)$.

A bounded positive integral operator Q from a measurable space E into a measurable space E' is an operator $f \mapsto Q(f)$ from $\mathcal{B}(E)$ to $\mathcal{B}(E')$ such that the functions

$$x \mapsto Q(f)(x) = \int_{E'} Q(x, \mathrm{d}y) f(y)$$

are bounded and measurable for some measure $Q(x, \cdot) \in \mathcal{M}(E')$. If $Q(1)(x) = 1$ for any $x \in E$, then Q is referred to as a Markov kernel from E to E'.

Let $G : x \in E \mapsto G(x) \in (0, \infty)$ be a bounded positive potential function. The following change of probability measures is referred to as Boltzmann-Gibbs transformation [9]:

$$\Psi_G : \eta \in \mathcal{M}(E) \mapsto \Psi_G(\eta) \in \mathcal{P}(E) \tag{1}$$

where, assuming $\eta(G) > 0$, $\Psi_G(\eta)(\mathrm{d}x) = \frac{1}{\eta(G)} G(x) \eta(\mathrm{d}x)$.

Also, the function mapping tuples to the associated sets is denoted $\mathscr{F}$ [1]. This function is useful since it allows set operations to be directly applied. In the next section, the concept of distinguishability is introduced for point processes and a solution to the problem of data association is deduced.

II. REPRESENTATION OF STOCHASTIC POPULATIONS

A. *Distinguishable and indistinguishable individuals*

Let $\mathscr{X}$ be a stochastic population described in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We represent the individuals of the population in the space $E_{\mathrm{s}} = \phi_{\mathrm{s}} \cup \mathbf{X}$, where $\mathbf{X}$ is a complete separable metric space, and where ϕ_{s} represents the case where a given individual does not have any image in $\mathbf{X}$. Based on E_{s}, the space of distinguishable and indistinguishable sets of individuals can be expressed as:

$$E = \bigcup_{i,j \geq 0} E_{\mathrm{s}}^i \times (E_{\mathrm{s}}^j / \sim), \tag{2}$$

where $\sim$ is an equivalence relation defined as follows: for any $n \in \mathbb{N} \setminus \{0\}$, and for any $x_1, x_2 \in E_{\mathrm{s}}^n$, $x_1 \sim x_2$ if and only if

$$\exists \sigma \in S_n, (x_{1,1}, \ldots, x_{1,n}) = (x_{2,\sigma(1)}, \ldots, x_{2,\sigma(n)}),$$

with S_n the symmetric group on n letters.

The subspace E_{s}^n (resp. $E_{\mathrm{s}}^n / \sim$) is the space of n distinguishable (resp. indistinguishable) individuals. More sophisticated spaces can be introduced, however, the focus of this paper is on independent populations so that E is sufficiently

[1] This "function" can be properly defined as a functor in category theory.

general to represent the stochastic population $\mathscr{X}$. The object of interest is the point process Φ defined as a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E, \mathcal{E})$. Note that the probability measure $p \in \mathcal{P}(E_\mathrm{s}^n)$ induced by Φ on E_s^n can be non-symmetric while the probability measure $\bar{p} \in \mathcal{P}(E_\mathrm{s}^n/\sim)$ induced by Φ on $E_\mathrm{s}^n/\sim$ is defined on equivalence classes and is therefore understood as symmetric.

B. The data association problem

The observation of $\mathscr{X}$ is represented by a point process Φ' from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E', \mathcal{E}')$, where E' is defined as

$$E' = \bigcup_{i,j \geq 0} E_\mathrm{s}'^i \times (E_\mathrm{s}'^j/\sim), \tag{3}$$

where E'_s is the individual space defined as $E'_\mathrm{s} = \phi'_\mathrm{s} \cup \mathbf{Z}$, with $\mathbf{Z}$ a complete separable metric space. The objective is to understand how an event in E' relates to events in E. Note that in a realisation of Φ', the only indistinguishable points in E' are the unobserved individuals of $\mathscr{X}$, therefore, any indistinguishable point $\bar{z} \in E'$ verifies $\bar{z} = \phi'_\mathrm{s}$.

The data association problem can be formulated in the product space $E \times E'$. However, when assuming that there is a one-to-one correspondence between individuals in E and observations in E', we can alternatively study data association in the subspace

$$\begin{aligned} E_\diamond = \bigcup_{i_1,\dots,i_4 \geq 0} & \big(E_\mathrm{s} \times E'_\mathrm{s}\big)^{i_1} \times \big((E_\mathrm{s} \times E'_\mathrm{s})^{i_2}/(O,\sim)\big) \\ & \times \big((E_\mathrm{s} \times E'_\mathrm{s})^{i_3}/(\sim, O)\big) \times \big((E_\mathrm{s} \times E'_\mathrm{s})^{i_4}/(\sim,\sim)\big), \end{aligned} \tag{4}$$

where O is the least equivalence relation on E_s^n and $E_\mathrm{s}'^n$ for any $n \in \mathbb{N} \setminus \{0\}$. For instance, the equivalence relation defined as $\sim_r = (O, \sim)$ on $(E_\mathrm{s} \times E'_\mathrm{s})^n$ is such that for all sequences $y, \hat{y} \in (E_\mathrm{s} \times E'_\mathrm{s})^n$, it holds that $y \sim_r \hat{y}$ if and only if there exists $\sigma \in S_n$ for which

$$\big((x_1, z_1), \dots, (x_n, z_n)\big) = \big((\hat{x}_1, \hat{z}_{\sigma(1)}), \dots, (\hat{x}_n, \hat{z}_{\sigma(n)})\big).$$

After endowing $E_\diamond$ with the σ-algebra $\mathcal{E}_\diamond$, one can define a point process $\Phi_\diamond$ as a measurable mapping from $(\Omega, \mathcal{F}, \mathbb{P})$ to the measurable space $(E_\diamond, \mathcal{E}_\diamond)$. Four different subspaces appear in the expression of $E_\diamond$, hence it is useful to study the behaviour of joint events on these subspaces separately before combining the results to solve the general problem.

1) Observed distinguishable individuals: Consider the event $y \in B_{\lhd\rhd} = (x \in B) \wedge (z \in B')$, where $\wedge$ is the *logical and*, $B \in \mathcal{E}_\mathrm{s}^{\otimes n}$, with $\otimes$ the σ-algebra product, and $B' \in \mathcal{E}_\mathrm{s}'^{\otimes n}$, defined as

$$y \in B_{\lhd\rhd} = (x_1 \in B_1, \dots, x_n \in B_n) \wedge (z_1 \in B'_1, \dots, z_n \in B'_n),$$

where the x_i's and the z_i's are distinguishable points. The event $y \in B_{\lhd\rhd}$ is equivalent to a specific event in $E_\diamond$:

$$y \in B_{\lhd\rhd} \Leftrightarrow \bigvee_{\sigma \in S_n} \bigwedge_{i=1}^{n} (x_i, z_{\sigma(i)}) \in B_i \times B'_{\sigma(i)} \tag{5}$$

where the notation $(x_i, z_{\sigma(i)})$ implies that x_i and $z_{\sigma(i)}$ represent the same individual and where $\vee$ is the *logical or*. We assume that two different points of a point process cannot represent the same individual so that the events in (5) are mutually exclusive. Denoting $X = \mathscr{F}(B)$ and $X' = \mathscr{F}(B')$, the joint probability measure $p_{\lhd\rhd} \in \mathcal{P}(E \times E')$, induced by Φ and Φ', can be expressed in terms of the probability measure $p_\diamond \in \mathcal{P}(E_\diamond)$, induced by $\Phi_\diamond$, as

$$\begin{aligned} p_{\lhd\rhd}(B_{\lhd\rhd}) &= p_\diamond(T_\mathrm{as}(X, X')) \\ &= \sum_{\sigma \in S_n} p_\diamond\big((B_1 \times B'_{\sigma(1)}) \times \dots \times (B_n \times B'_{\sigma(n)})\big), \end{aligned}$$

where the mapping T_as, with "as" for "association", corresponds to the equivalence (5) but for measurable subsets.

2) Observed indistinguishable individuals: Consider the event $y \in B_{\lhd\rhd} = (\bar{x} \in \bar{B}) \wedge (z \in B')$ with $B \in \bar{\mathcal{E}}_\mathrm{s}^{\otimes n}$, the σ-algebra on $E_\mathrm{s}^n/\sim$, and $B' \in \mathcal{E}_\mathrm{s}'^{\otimes n}$, defined as

$$y \in B_{\lhd\rhd} = (\bar{x}_1 \in \bar{B}_1, \dots, \bar{x}_n \in \bar{B}_n) \wedge (z_1 \in B'_1, \dots, z_n \in B'_n),$$

where the $\bar{x}_i$'s represent indistinguishable individuals. The event $y \in B_{\lhd\rhd}$ is equivalent to a specific event in $E_\diamond$:

$$y \in B_{\lhd\rhd} \Leftrightarrow \bigvee_{\sigma \in S_n} \bigwedge_{i=1}^{n} (\bar{x}_i, z_{\sigma(i)}) \in \bar{B}_i \times B'_{\sigma(i)}, \tag{6}$$

where there is no identification between $\bar{x}_i$ and $z_{\sigma(i)}$ since $\bar{x}_i$ does not represent a specific individual of the population $\mathscr{X}$. The events composing (6) are not mutually exclusive, however, when setting $\bar{B}_i = E_\mathrm{s}$, for any $1 \leq i \leq n$, they become all equivalent. In this case, denoting $X' = \mathscr{F}(B')$, the equivalence (6) can be expressed as a mapping T_fa, with "fa" for "false alarm", as follows:

$$T_\mathrm{fa}(X') = (E_\mathrm{s} \times B'_1) \times \dots \times (E_\mathrm{s} \times B'_n).$$

The choice $\bar{B}_i = E_\mathrm{s}$ is justified in practice since the interest lies in the probability for a given observation to be a false alarm or an appearing individual; and the localisation in E_s does not help because this information cannot be specific.

3) Unobserved distinguishable and indistinguishable individuals: Consider the event $y \in B_{\lhd\rhd} = (x \in B) \wedge (\bar{z} \in \bar{B}')$, with $B \in \mathcal{E}_\mathrm{s}^{\otimes n}$ and $\bar{B}' \in \bar{\mathcal{E}}_\mathrm{s}'^{\otimes n}$, defined as

$$y \in B_{\lhd\rhd} = (x_1 \in B_1, \dots, x_n \in B_n) \wedge (\bar{z}_1 \in \bar{B}'_1, \dots, \bar{z}_n \in \bar{B}'_n).$$

In this case, we know that $\bar{z}_1 = \dots = \bar{z}_n = \phi'_\mathrm{s}$ with probability 1, so that denoting $X = \mathscr{F}(B)$, the function T_md corresponding to the previous equivalence can be written

$$T_\mathrm{md}(X) = (B_1 \times \phi'_\mathrm{s}) \times \dots \times (B_n \times \phi'_\mathrm{s}),$$

where "md" stands for "missed-detection". The same result holds when $B \in \bar{\mathcal{E}}_\mathrm{s}^{\otimes n}$ and when $B_i = E_\mathrm{s}$ for any $1 \leq i \leq n$, then the corresponding mapping is denoted T_r and defined as $T_\mathrm{r}(n) = (E_\mathrm{s} \times \phi'_\mathrm{s})^n$, where the subscript "r" is for "remainder".

4) General association: Before demonstrating the main result, additional notations are required for the sake of compactness. Let X and X' be two sets of the same size, then $S_{X,X'}$ is the set of all bijections between X and X' and the function F is defined as

$$F(X, X') = \bigcup_{f \in S_{X,X'}} \Big(\big(y, f(y)\big)\Big)_{y \in X}. \tag{7}$$

Proposition 1: Let X and X' be two sets of the same size and let $X' = X'_1 \cup X'_2$. If $X'_1 \cap X'_2 = \emptyset$, then

$$F(X, X') = \bigcup_{\hat{X} \subseteq X : |\hat{X}| = |X'_1|} \big(F(\hat{X}, X'_1), F(\hat{X}^c, X'_2)\big).$$

This proposition states that the union over all possible associations between X and X' can be reformulated as the union over all possible ways of distributing elements of X among X'_1 and X'_2 and then considering the union over all possible associations in the corresponding subsets. In the following theorem, the results of the previous sections are combined to solve the general data association problem.

Theorem 1 (Data Association): Let $B_{\diamond}$ be a measurable subset in $\mathcal{E} \otimes \mathcal{E}'$ with $B_{\diamond} = (B \times \bar{B}) \times (B' \times \bar{B}')$ such that

$$B \in \mathcal{E}_{\mathrm{s}}^{\otimes n}, \quad \bar{B} = E_{\mathrm{s}}^{\bar{n}}, \quad B' \in \mathcal{E}_{\mathrm{s}}'^{\otimes k}, \quad \bar{B}' = \phi_{\mathrm{s}}'^{\bar{k}},$$

where $k + \bar{k} = n + \bar{n}$, and let $X = \mathscr{F}(B)$ and $X' = \mathscr{F}(B')$. The measurable subset $B_{\diamond}$ can be expressed in $\mathcal{E}_{\diamond}$, via X and X', through the following mapping:

$$T^*_{\diamond}(X, X', \bar{n}) = \bigcup_{\substack{Y \subseteq X, Y' \subseteq X' \\ |Y| = |Y'|}} T_{\mathrm{as}}(Y, Y') \times T_{\mathrm{fa}}(Y'^c) \times T_{\mathrm{md}}(Y^c) \times T_{\mathrm{r}}(\bar{n} - |Y'^c|),$$

and we additionally define $T_{\diamond}(B_{\diamond}) = T^*_{\diamond}(X, X', \bar{n})$.

Proof: Let $\bar{X} = \mathscr{F}(\bar{B})$, $\bar{X}' = \mathscr{F}(\bar{B}')$ and $U = X \cup \bar{X}$. Applying Proposition 1 on U and $X' \cup \bar{X}'$ gives

$$F(U, X' \cup \bar{X}') = \bigcup_{\hat{X} \subseteq U : |\hat{X}| = k} F(\hat{X}, X') \times F(\hat{X}^c, \bar{X}').$$

The set $\hat{X}$ is composed of elements from both X and $\bar{X}$ so that, denoting $Y = \hat{X} \cap X$ and $\bar{Y} = \hat{X} \cap \bar{X}$ and noting that $\hat{X}^c = Y^c \cup \bar{Y}^c$ when taking the complement of Y and $\bar{Y}$ in X and $\bar{X}$ respectively, we can write

$$F(U, X' \cup \bar{X}') = \bigcup_{\substack{Y \subseteq X, \bar{Y} \subseteq \bar{X} \\ |Y| + |\bar{Y}| = k}} F(Y \cup \bar{Y}, X') \times F(Y^c \cup \bar{Y}^c, \bar{X}'),$$

it is thus possible to further develop the expression as follows:

$$F\big(Y \cup \bar{Y}, X'\big) = \bigcup_{Y' \subseteq X' : |Y'| = |Y|} F(Y, Y') \times F(\bar{Y}, Y'^c),$$

$$F(Y^c \cup \bar{Y}^c, \bar{X}') = \bigcup_{\bar{Y}' \subseteq \bar{X}' : |\bar{Y}'| = |Y^c|} F(Y^c, \bar{Y}') \times F(\bar{Y}^c, \bar{Y}'^c).$$

However, the choices $\bar{Y} \subseteq \bar{X}$ and $\bar{Y}' \subseteq \bar{X}'$ are irrelevant and it holds that $|\bar{Y}| = k - |Y'|$ and $|\bar{Y}'| = n - |Y|$ so that the overall union simplifies to a union over $\{Y \subseteq X, Y' \subseteq X' : |Y| = |Y'|\}$. Finally, the following identifications allow to prove the desired result: $F(X \cup \bar{X}, X' \cup \bar{X}') = T^*_{\diamond}(X, X', |\bar{X}|)$ as well as $F(Y, Y') = T_{\mathrm{as}}(Y, Y')$, $F(\bar{Y}, Y'^c) = T_{\mathrm{fa}}(Y'^c)$, $F(Y^c, \bar{Y}') = T_{\mathrm{md}}(Y^c)$ and $F(\bar{Y}^c, \bar{Y}'^c) = T_{\mathrm{r}}(|\bar{Y}^c|)$. ∎

Theorem 1 can be interpreted as follows: a given observation z can be either associated with a distinguishable individual or understood as a false alarm or an appearing individual. Also, a given distinguishable individual can be either detected or missed-detected. Even though this result is intuitive, we have seen that the introduction of distinguishability allows for demonstrating it using only simple probability rules.

III. Association measure

In this section, the objective is to detail and justify the form of the prediction and correction steps for a representation of the independent stochastic population $\mathscr{X}$.

The random (finite) observation set at time t is denoted Z_t and $\mathcal{Z}_t$ is the observation random measure at time t, defined as

$$\mathcal{Z}_t = \sum_{z \in Z_t} \delta_z,$$

where δ_z is the Dirac measure at z. Additionally, we denote $Z^{\mathrm{s}}_t = Z_t \cup \phi'_{\mathrm{s}}$ and $\mathcal{Z}^{\mathrm{s}}_t = \mathcal{Z}_t + \delta_{\phi'_{\mathrm{s}}}$ the observation set and random measure including the empty observation ϕ'_{s}. Also, let the set Y^{s}_t of sequences of observations be defined as

$$Y^{\mathrm{s}}_t = \Big\{(z_0, \dots, z_t) : z_{t'} \in Z^{\mathrm{s}}_{t'}, 0 \le t' \le t\Big\},$$

and let $y_{\mathrm{s},t} \in Y^{\mathrm{s}}_t$ be such that $z_{t'} = \phi'_{\mathrm{s}}$, $0 \le t' \le t$. Elements in Y^{s}_t are called observation paths up to time t and we define $Y_t = Y^{\mathrm{s}}_t \setminus y_{\mathrm{s},t}$.

Let $\hat{\mathcal{A}}_t$ (resp. $\mathcal{A}_t$) be the updated (resp. predicted) association measure at time t, describing the probability of the observation paths in Y_t (resp. Y_{t-1}), defined as

$$\hat{\mathcal{A}}_t = \sum_{y \in Y_t} \hat{a}_y \delta_y \quad \text{and} \quad \mathcal{A}_t = \sum_{y \in Y_{t-1}} a_y \delta_y,$$

where $a_y, \hat{a}_y \in [0,1]$ for any $y \in Y_t$. Furthermore, we define $\mathcal{A}^{\mathrm{s}}_t = \mathcal{A}_t + \delta_{y_{\mathrm{s},t-1}}$. Note that $a_{y_{\mathrm{s},t}} = 1$ at any time t.

Let $\hat{p}^{(y)}_t \in \mathcal{P}(\mathbf{X})$ (resp. $p^{(y)}_t \in \mathcal{P}(\mathbf{X})$) be the law of the updated (resp. predicted) state of the hypothesis with observation path $y \in Y_t$ (resp. $y \in Y_{t-1}$) at time t. The distribution $p^{(y_{\mathrm{s},t-1})}_t \in \mathcal{P}(E_{\mathrm{s}})$ is the (common) law of the indistinguishable individuals in the stochastic population $\mathscr{X}$ at time t, which, by definition, have never been observed.

A. Initialisation

At time $t = 0$, no observation has been made available yet so that no individual can be distinguished. Thus, the measure $\gamma_0 \in \mathcal{M}(\mathbf{X})$ representing the initial state of the distinguishable individuals is set to $\gamma_0 = 0$.

B. Prediction

Let $\hat{\gamma}_t \in \mathcal{M}(\mathbf{X})$ be the measure representing the state of the distinguishable individuals at time t, defined as

$$\hat{\gamma}_t = \int \hat{\mathcal{A}}_t(\mathrm{d}y)\hat{p}^{(y)}_t.$$

Since all the individuals in the population are assumed to be independent, the measure $\hat{\gamma}_t$ can be predicted straightforwardly. Let $f \in \mathcal{B}(E_{\mathrm{s}})$ and let $M_{t,t+1}$ be a Markov kernel from $\mathbf{X}$ to E_{s}, then, for any $y \in Y_t$,

$$\mathcal{A}_{t+1}(\mathrm{d}y)p^{(y)}_{t+1}(f) = \hat{\mathcal{A}}_t(\mathrm{d}y)\hat{p}^{(y)}_t(M_{t,t+1}(\mathbf{1}_{\mathbf{X}}f)),$$

so that

$$\gamma_{t+1} = \int \mathcal{A}_{t+1}(\mathrm{d}y)p^{(y)}_{t+1},$$

and where $\hat{p}^{(y)}_t(M_{t,t+1}(\mathbf{1}_{\mathbf{X}}))$ can be interpreted as the probability for the hypothesis y to survive to time $t+1$.

C. Correction

There are different ways of formulating Bayes' theorem for the distributions $p_t^{(\cdot)}$. The first way is to only consider the joint probability of an hypothesis $y \in Y_{t-1}$ together with an observation $z \in Z_t$. When $G(x) = g_t(z, x)$ is a (bounded positive) likelihood function on $E'_\mathrm{s} \times E_\mathrm{s}$, Bayes' theorem can be expressed by using (1), with $f \in \mathcal{B}(\mathbf{X})$, as

$$\hat{p}_t^{(y,z)}(f) = \frac{p_t^{(y)}(f\, g_t(z,\cdot))}{p_t^{(y)}(g_t(z,\cdot))} = \Psi_{g_t(z,\cdot)}(p_t^{(y)})(f). \tag{8}$$

Although (8) provides the corrected distribution $\hat{p}_t^{(y,z)}$, it does not assess the probability for y and z to be actually associated. Estimating this probability requires the full population to be involved, as demonstrated in the following theorem.

Theorem 2: Let $\bar{n}$ be the number of indistinguishable individuals. For a specific observation $z \in Z_t$, the probability of the association between y and z, for any $y \in Y_{t-1}^\mathrm{s}$, is

$$\hat{p}_{t,z}^{(y,z)} = a_y \hat{q}_{t,z}^{(y,z)} = \frac{a_y p_t^{(y)}\big(\mathbf{1}_\mathbf{X} g_t(z,\cdot)\big) w(y,z,\bar{n})}{\int \mathcal{A}_t^\mathrm{s}(\mathrm{d}y') p_t^{(y')}(g_t(z,\cdot)) w(y',z,\bar{n})}, \tag{9}$$

where $\mathbf{1}_\mathbf{X}$ discards the false alarm term in the numerator and where w also depends on $\mathcal{A}_t$ and Z_t. For a given $y \in Y_{t-1}$ and any $z \in Z_t^\mathrm{s}$, the probability of associating y and z is

$$\hat{p}_{t,y}^{(y,z)} = a_y \hat{q}_{t,y}^{(y,z)} = \frac{a_y p_t^{(y)}(g_t(z,\cdot)) w(y,z,\bar{n})}{\int \mathcal{Z}_t^\mathrm{s}(\mathrm{d}z') p_t^{(y)}(g_t(z',\cdot)) w(y,z',\bar{n})}. \tag{10}$$

The term $w(y, z, \bar{n})$ is interpreted as the joint probability of all the hypotheses except y and all the observations except z.

Proof: Using the result and notations of Theorem 1 with subsets in $\mathcal{E}_\mathrm{s}^{\otimes n}$ indexed by $y \in Y_{t-1}$, so that $n = |Y_{t-1}|$, and subsets in $\mathcal{E}_\mathrm{s}'^{\otimes k}$ indexed by $z \in Z_t$, so that $k = |Z_t|$, we see that the global data association problem can be rewritten in different ways: either, for any $z \in Z_t$, as

$$\begin{aligned} T_\diamond(B_{\triangleleft\triangleright}) = &\bigcup_{y \in Y_{t-1}} (B_y \times B_z) \times T_\diamond^*\big(X \setminus B_y, X' \setminus B_z, \bar{n}\big) \\ &\cup (E_\mathrm{s} \times B_z) \times T_\diamond^*\big(X, X' \setminus B_z, \bar{n}-1\big), \end{aligned} \tag{11}$$

or, for any $y \in Y_{t-1}$, as

$$\begin{aligned} T_\diamond(B_{\triangleleft\triangleright}) = &\bigcup_{z \in Z_t} (B_y \times B_z) \times T_\diamond^*\big(X \setminus B_y, X' \setminus B_z, \bar{n}\big) \\ &\cup (B_y \times \phi'_\mathrm{s}) \times T_\diamond^*\big(X \setminus B_y, X', \bar{n}\big). \end{aligned} \tag{12}$$

Setting $B_y = E_\mathrm{s}$ for any $y \in Y_{t-1}$, we define:

$$\begin{aligned} w(y,z,\bar{n}) &= p_\diamond^{(\cdot)}\big(T_\diamond^*(X \setminus B_y, X' \setminus B_z, \bar{n})\big), \\ w(y_{\mathrm{s},t-1}, z, \bar{n}) &= p_\diamond^{(\cdot)}\big(T_\diamond^*(X, X' \setminus B_z, \bar{n}-1)\big), \\ w(y, \phi'_\mathrm{s}, \bar{n}) &= p_\diamond^{(\cdot)}\big(T_\diamond^*(X \setminus B_y, X', \bar{n})\big), \end{aligned}$$

where $p_\diamond^{(\cdot)}$ are projections of $p_\diamond$. Then, denoting $p_\diamond^{(y,z)}$ the projection of $p_\diamond$ on $B_y \times B_z$ and assuming that $p_\diamond^{(y,z)}(B_y, \cdot)$ is absolutely continuous w.r.t. a reference measure in E'_s, we write $p_\diamond^{(y,z)}(B_y, z) = a_y p_t^{(y)}(g_t(z,\cdot))$. Thus (9) (resp. (10)) follows from Bayes' theorem with one element of the union (11) (resp. (12)) as numerator and the union as denominator. ■

Based on the result of Theorem 2, the corrected association measure $\hat{\mathcal{A}}_t$ can be expressed in two different ways:

$$\begin{aligned} \hat{\mathcal{A}}_t(\mathrm{d}(y,z)) &= \mathcal{A}_t^\mathrm{s}(\mathrm{d}y)\mathcal{Z}_t(\mathrm{d}z)\hat{q}_{t,z}^{(y,z)} + \mathcal{A}_t(\mathrm{d}y)\delta_{\phi'_\mathrm{s}}(\mathrm{d}z)\hat{q}_{t,y}^{(y,z)}, \\ &= \mathcal{A}_t(\mathrm{d}y)\mathcal{Z}_t^\mathrm{s}(\mathrm{d}z)\hat{q}_{t,y}^{(y,z)} + \delta_{y_{\mathrm{s},t-1}}(\mathrm{d}y)\mathcal{Z}_t(\mathrm{d}z)\hat{p}_{t,z}^{(y,z)}. \end{aligned}$$

The corrected measure $\hat{\gamma}_t \in \mathcal{M}(\mathbf{X})$, representing the population state, can then be expressed as

$$\hat{\gamma}_t = \int \hat{\mathcal{A}}_t(\mathrm{d}y)\hat{p}_t^{(y)} = \int \hat{\mathcal{A}}_t(\mathrm{d}y)\Psi_{g_t(z,\cdot)}(p_t^{(y)}). \tag{13}$$

Note that, unlike the PHD filter [2], the corrected measure $\hat{\gamma}_t$ cannot be expressed in terms of the predicted measure γ_t only, because of the dependence of w on y. It has to be underlined that the approach presented here does not involve strong approximations on the predicted law of the population, such as Poisson i.i.d., but only assumes that objects are statistically independent and that no more than one measurement is originated from each of them. However, the complexity of the terms $w(y, z, \bar{n})$ makes (9) and (10) difficult to compute. Therefore, to use this approach in practice, one has to find an approximation for these terms. An accurate approximation will give an accurate filter but will be computationally intensive while a stronger approximation can alleviate the computational cost. Another useful aspect of (10) is that the probability for an hypothesis to be missed-detected can also be updated through Bayes' theorem, giving an a posteriori probability of missed-detection.

IV. Conclusion

The concept of distinguishability in the representation of a stochastic population has been proved useful to the derivation of a data association scheme for independent individuals. This approach has been used in practice to define an association measure for the estimation of independent stochastic populations. It has been demonstrated that this association measure can be expressed in two different ways, thus offering a choice in the representation of the association for future exploitation.

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